APPENDIX A-2: ROUNDING

Rounding – Using a neighboring number of a particular precision

C. **Round Numbers:**

Although numbers can be used to express very small subdivisions (by the use of long decimal fractions, for example), it is often useful to limit the numbers used to ones that are multiples of some unit that is just small enough that a step of that size is not very important. This process is called ***rounding***, and the size of the step is called the ***rounding precision***.

When rounding, each number except the exact multiples of the rounding precision is changed into one of the two neighboring multiples that are just larger or smaller than it. Usually the **nearest neighbor** is used, but see section D for a full discussion of the various rounding methods. The difference between the original number and the rounded value is called the ***rounding error***.

(a) **Rounding to whole numbers** is particularly common (why worry about fractions when you don't need to?). This corresponds to a rounding precision of 1. If no precision is stated, it is assumed that the term "rounding" means replacing the original number with a whole number.

(b) **Rounding to powers of 10** is also frequently done (to hundreds or millions, for example). In such cases, either a term such as "millions" is used ("The population of France is 61 million"), or the number is expressed using zeros ("France has about 61,000,000 people") whose purpose is just to show the size of the number, not to mean that an exact number is claimed. These extra "non-significant" zeros can be ambiguous, since it is not certain what the precision is. A report that Austin population is 490,000 is obviously using a rounded figure, for example, but is the rounding precision 1,000 or 10,000? The methods that have been developed to avoid confusions like this will be shown later in this course.

(c) **Rounding to a particular number of decimal places** is also very common. Baseball batting averages (always calculated to 3 decimal places, such as .235) are an example. Since the rounding precision is the smallest step between neighboring rounded values, it is 0.001 in this case (in other cases, it might be 0.1 or 0.00001, depending on the number of decimal places).

Significant zeros: When this type of rounding is being used, any zeros at the end of the rounded number are kept in order to show the rounding precision. Thus a 30-out-of-100 batting average is expressed as 0.300, not as 0.3, even though these would be considered the same if they were exact rather than rounded numbers. This is similar to the way that $35.10 is written rather than $35.1 – in fact, the use of cents in monetary calculations is an instance of 2-decimal-place rounding.

(d) **Rounding to a particular number of digits** (which is different than a specific number of decimal places, as explained below) is done when it is the ***relative precision*** (the precision compared to the number's size) that is important. For example, "rounding to three-digit accuracy", where the first three non-zero digits are retained, will always have a rounding error of less than 1%, since the number is from 100 to 999 times larger than the precision. Seven-digit accuracy, which is typical of "single-precision floating-point" numbers in computers, has a rounding error of no more than 1 part in a million.

The contrast between these different ways of looking at precision can be shown by some examples. The value 137.04 has *five-digit accuracy* but has *two-decimal-place precision*. On the other hand, a small decimal fraction like 0.00012 has the opposite: *two-digit accuracy* but *five-decimal-place precision*. Which is more important depends on how the number is going to be used. In general, number-of-decimal-place precision is more important for addition and subtraction, while number-of-digit accuracy is more important for multiplication and division. This will be explained in greater detail in the discussion of how to estimate the errors that are likely to result when approximate numbers are used in different kinds of calculations.

(e) **Rounding to multiples of numbers not related to ten** is also possible, although much less common. For example, a store might round its bill to multiples of 5 cents, to avoid the need to use pennies. Also, sometimes values are rounded to multiples of simple fractions such as ¼, or to fractional parts of 100, such as 25 or 50.

Problems:

[5] *Express the value of the following numbers or expressions to the nearest neighboring rounded value, using the rounding precision given in each case:*

(a) rounded to a whole number: 43.19 -25.701 8.5

(b) rounded to five-digit accuracy: $^{22}/_{7}$ 89,685,173 tangent of 85°

(c) rounded to three decimal places: 0.00148 $^{2}/_{3}$ sine of 30°

(d) rounded to millions: 89,685,173 490,582

[6] *For each rounded number listed below, state the apparent rounding precision (for example, the apparent rounding precision of 91.7 is 0.1).*

(a) 2.72 (b) 133,000 (c) 9.2 million (d) 0.022 (e) 12 (f) 0.00350

[7] *For each item in question [6], state the number of apparent number of digits of accuracy (for example, 91.7 is stated with an apparent accuracy of 3 digits).*

D. **Rounding Methods:**

Unless the number to be rounded is already exactly equal to one of the neighboring values that match the desired rounding precision, a ***rounding rule*** is used to select which neighbor to use:

(a) **nearest-neighbor rounding** ***is the standard method***, with the closer rounded number used (for the rare cases exactly halfway between the neighboring rounded values, a special rule* is used if maximum accuracy is needed). For nearest-neighbor rounding, the original value differs from the rounded value by no more than ½ the precision, and the rounding error (the original value subtracted from the rounded value) may be either positive or negative. **This form of rounding is the most accurate and is normally used with measurements.** *Note that when the nearer neighbor is the one that is further from zero than the original number, the final digit of the rounded number will be different than the digit in the same place in the original number.*

(b) ****tie-breaking rules*:** If the exactly-halfway-between numbers are always rounded in the same direction, the rounding process will be slightly *biased*, and will, if many numbers fall on the halfway mark, tend to incorrectly increase (or decrease, depending on the preferred direction) the sum of the rounded numbers. People such as accountants who want to do particularly exact work have long used a special rule to avoid this bias. They round such exactly-halfway cases to the nearest ***even*** number (thus 8.5 is rounded to 8 rather than 9 – but note that 7.5 is also rounded to 8, rather than to 7). The idea is that in a group of such halfway-between numbers, how many are being rounded *up* will turn out to be roughly the same as how many are being rounded *down*, so the final total will be closer to correct (what it would have been without rounding) than if the biased method had been used. Calculators and computers usually work with so many digits of precision that they do not use this rule, and it is thus no longer as important as it once was, especially if rounding is done only at the very end of the calculation process, as computers and calculators make it easier to do.

(c) What about "**rounding down**" and "**rounding up**"? There are actually four different methods in which all the values between two particular neighboring rounded values are always rounded to only one of those values. The methods are:

[i] **truncate** – The *truncate* function drops the fractional part that would remain after dividing by the precision. Thus in truncation rounding to whole numbers, 3.9 would be rounded to 3 rather than 4, even though 4 is nearer. This method is convenient because no digits ever have to be changed, but truncation can introduce significant bias. While use of the truncate method is sometimes informally called "rounding down", truncation of a negative number, such as the conversion of –2.6 to –2, actually makes it a higher (that is, less negative) value and thus is really "rounding up", at least as far as addition is concerned. Rather

than “rounding down”, a more correct way to describe *truncate* would be “rounding toward zero”.

[ii] **<u>floor</u>** – The *floor* function rounds to the closest rounded value that ***does not exceed the number***. Thus in rounding to whole numbers, *floor*(5.2) equals 5, while *floor*(–5.2) equals –6. Thus the *floor* function really does always “round down”, unless the number is already exactly equal to the rounded value. For positive numbers, *floor* and *truncate* give the same result.

[iii] **<u>ceiling</u>** – The *ceiling* function rounds to the closest rounded value that ***the number does not exceed***. It thus always “rounds up” (unless the number was already exactly equal to the rounded value), converting 7.2 to 8 when rounding to whole numbers. Note that *ceiling*(–7.2) equals –7, illustrating that for negative numbers, the *ceiling* function produces the same result as *truncate*.

[iv] The remaining method, **<u>rounding away from zero</u>** (the opposite of *truncate*), does not have a standard short name because it is not often used. It can be computed by using the *ceiling* function for positive numbers and the *floor* function for negative numbers. Note that the often-used tie-breaking method that always rounds up for positive halves and down for negative halves (4.5 is rounded to 5, and –4.5 is rounded to –5) is using the rounding-away-from-zero procedure, although only for values exactly on the halfway mark.

Problems:

[8] *Give three examples of practical applications where rounding rules are used that are not standard nearest-neighbor rounding. Try to find examples that use different rules.*

[9] *In the table below, fill in the spaces so that the numbers on the left are rounded according to the different rules at the top of the columns, using the stated precision or accuracy.*

Values	***precision***	**Nearest neighbor**	**Trun-cate**	**Floor**	**Ceiling**	**away from zero**
3.1415935	.0001					
-459.67	1					
20,851,820	millions					
.0008375	5 decimal places					
125	10					
186,282.397	5 digits					
-273.15	1					