APPENDIX A-3: SCIENTIFIC & ENGINEERING NOTATION; UNITS

Scientific Notation, Engineering Notation, and Measurement Units

Total number of atoms forming the planet Earth:
Approximately 64,500,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000
or 6.45×10^{49}

Weight of one molecule of water, in pounds:
approximately 0.00000000000000000000000006576 ***or*** 6.576×10^{-26}

Those working in science or technology encounter some very large and very small numbers. A pound of iron, for example, contains close to 4,901,000,000,000,000,000 atoms. The lethal dose of botulism toxin is about 0.00000005 grams. The wavelength of the yellow light of a sodium streetlamp is 0.000000589 meters. Numbers expressed in this fashion are difficult to use – all the extra zeros are both tedious and confusing.

People who continually work with numbers of very different sizes have developed methods to deal with these problems and make their work easier. These methods also make it easy to ensure that numbers can be expressed compactly with no need for non-significant zeros before or after the numbers that actually contain the information.

There are two main ideas. The first is **exponential notation** to permit numbers of any size to be expressed with equal ease. In its simplest form, based on **powers of 10**, this is called ***scientific notation***, which completely avoids superfluous zeros.

The other idea is a set of **standard multiplier prefixes for measurement units** to make most measurements even easier to express. Each standard multiplier differs from the next one by a factor of 1000 (such as seconds-***milli***seconds-***micro***seconds-***nano***seconds for smaller units or bytes-***kilo***bytes-***mega***bytes-***giga***bytes for larger ones). The ideas of exponential notation and factor-of-1000 steps are combined in ***engineering notation***, which makes it easy to switch back and forth between exponential and units-based notation as needed.

Powers of Ten ("orders of magnitude")

between one-millionth (10^{-6}) and one million (10^{6})

10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}	10^{4}	10^{5}	10^{6}
0.000001	0.00001	0.0001	0.001	0.01	0.1	1	10	100	1,000	10,000	100,000	1,000,000

Scientific Notation

In **scientific notation**, a quantity is always expressed as ***a number between 1 and 10, multiplied times an integer power of 10***. (If the number is negative, a minus sign is prefixed to it but otherwise positive and negative values are treated exactly the same.) The quantity 250,000, for example, is expressed in scientific notation as 2.5×10^5. The 2.5 is called the **mantissa** of the number and the 10^5 is called the **order of magnitude** of the number. The table above summarizes the relationship between orders of magnitude and ordinary notation.

The exponent in the order of magnitude (the 5 in the case of 10^5) will be positive for values that are 10 or larger, zero for values between 10 and 1, and negative for values between one and zero. (If a number is exactly zero, it is just expressed as 0 without any $\times 10^n$ term.) For example, the decimal fraction 0.015 would be expressed as 1.5×10^{-2}. The minus sign in the exponent means to multiply by the *reciprocal* value $^1/_{100}$, rather than $10^2 = 100$.

Notice that in scientific notation there is little difference in form between numbers reflecting the mass of the earth (5.976×10^{27} grams), the mass of an electron (9.1×10^{-28} grams), and the mass of a half-dollar (1.134×10^1 grams). This generality is an important advantage of scientific notation, although it would certainly be easier to express some values, like the 11.34 grams for the half-dollar, in regular notation.

For any particular value, there is only one correct way to express it in scientific notation. If the number is negative, it will start with a minus sign (and after that be expressed just like a positive number). There will be a **single non-zero digit to the left of the decimal point**. There can be as many digits to the right of the decimal point as needed to express the value, but these should only include digits that are being presented as significant. Thus 5.100×10^3 means that the value is within ½ of exactly 5100, while 5.1×10^3 means that the value is a rounded approximate number that is closer to 5100 than it is to 5000 or 5200, but is not promised to be closer than that. The complete absence of non-significant zeros is one of the main advantages of scientific notation.

"E" notation: Computers and calculators often use an alternative "E" notation to express the order of magnitude, displaying "2.4E5" or "2.4E+05" instead of 2.4×10^5, and "1.5E-2" or "1.5E-02" instead of 1.5×10^{-2}. In calculators, the order of magnitude is sometimes shown as two smaller raised numbers at the far right side of the numeric display. *All these representations mean the same thing as the* $\times 10^n$ *notation*: multiply the mantissa by the power of 10 indicated by the order-of-magnitude exponent.

A more significant way that calculators deviate from scientific notation is that they often show insignificant digits. Since they can't determine if a number is exact or approximate, any division or other operation that does not give an even answer will be expressed with all the decimal values that the calculator display will hold. Solving the question "*If an object that weighs about 14 pounds is divided into three equal pieces, how much does each piece weigh?*" on a calculator will give an over-precise 4.6666667, rather than the correct "about 4.7 pounds".

Converting from standard to scientific notation

Since 10 (or any other number) produces a value of 1 when raised to the power 0 (which means a number divided by itself), converting a number to scientific notation begins by multiplying the number by 10^0, which does not change its value. This puts the number into mantissa-and-power form. The mantissa and exponent are then simultaneously changed in a way that makes the mantissa a number between 1 and 10 without changing the overall value of the number. If the mantissa starts out too big, the decimal point is moved to the left at the same time that the exponent is added to, and becomes positive. If the mantissa starts out too small, the decimal point is moved to the right and the exponent is subtracted from, and becomes negative.

Thus to convert 490,000 (the approximate Austin population) to scientific notation, start with $\mathbf{490{,}000 \times 10^{0}}$. But the mantissa will have to be much smaller for it to comply with the between-1-and-10 rule. So start dividing the mantissa by 10, and balance this (so the overall value won't change) by multiplying the order-of-magnitude part by 10. This moves the decimal one place to the left in the mantissa, and increases the exponent by 1. This process is repeated until the mantissa is in the correct range. To illustrate, here is the full list of steps:

$\mathbf{490{,}000 \times 10^{0}}$ *– the mantissa is too large, and must be divided by ten*
$\mathbf{49{,}000 \times 10^{1}}$ *until it is the right size*
$\mathbf{4{,}900 \times 10^{2}}$ At each step, the decrease in size of the mantissa is
$\mathbf{490 \times 10^{3}}$ exactly balanced by the increase in the power of 10.
$\mathbf{49 \times 10^{4}}$
$\mathbf{4.9 \times 10^{5}}$ *– the mantissa is in the correct range,*
so this is scientific notation

A much faster way to get the same result is to just **count how far the decimal point must be moved to the left, then use that as the exponent** in the order of magnitude. If the starting number is less than one, so the decimal point must be moved to the right, then the exponent will be the corresponding negative number. In that case, each step involves the opposite of before: now *multiplying* the mantissa by 10, with a balancing *reduction* of the power of 10. For example, $\mathbf{0.015 \leftrightarrow 0.015 \times 10^{0} \leftrightarrow 0.15 \times 10^{-1} \leftrightarrow 1.5 \times 10^{-2}}$.

Potential disadvantages to scientific notation

While scientific notation has many useful features, it is most useful when either [i] a variety of numbers that are very different in size are being used (as in some areas of science, hence the name), or [ii] when the computation is being done automatically without looking at the intermediate results (as in the internal operations of computers and calculators, which mainly use a base-2 equivalent of scientific notation called *floating-point* representation).

In discussions and calculations by people working on a particular set of problems that use a narrower range of values, as is common in applied work, some disadvantages of scientific notation become evident.

One problem is that although rough "order of magnitude" comparisons are very easy with scientific notation (just compare the exponents), the use of different powers of ten (and often of different precisions) for each different number can make it difficult to quickly make close comparisons. 9.45×10^{5} and 1.1×10^{6} are close to each other, for example, while 9.4×10^{6} and 1.10×10^{5} differ substantially (convert them to ordinary form and see). Comparisons are much easier if the same exponents are used, since then only the mantissas need be compared.

The other main problem with scientific notation is also related to units – most powers of ten do not have standard units associated with them. If a computer computation takes 6×10^{-4} seconds, its duration can be talked about as either 600 microseconds or 0.6 milliseconds, but both of these require that the 6 in the mantissa be further converted, which wastes time and promotes confusion. It would be better if numbers being used in connection with unit prefixes were kept consistently in a units-compatible form. Engineering notation addresses this need.

Powers of 1000 between one-trillionth (10^{-12}) and one trillion (10^{12})

Power	Value	Common Name	Standard Prefix	Abbreviation
$\mathbf{10^{-12}}$	0.000000000001	trillionths	Pico-	p
$\mathbf{10^{-9}}$	0.000000001	billionths	Nano-	n
$\mathbf{10^{-6}}$	0.000001	millionths	Micro-	μ

10^{-3}	0.001	thousandths	Milli-	m
10^{0}	1	ones	*– no prefix –*	
10^{3}	1,000	thousands	Kilo-	k
10^{6}	1,000,000	millions	Mega-	M
10^{9}	1,000,000,000	billions	Giga-	G
10^{12}	1,000,000,000,000	trillions	Tera-	T

Engineering Notation

Numbers can often be more easily matched with standard units and measurement-device settings by using **engineering notation**. As shown below, the powers of ten used in engineering notation are all multiples of 3. The rule limiting the integer part of the mantissa to *from 1 to 9* is changed to *from 1 to 999*, to allow all numbers to be expressed. Thus the scientific-notation value of 6×10^{-4} seconds referred to in the previous section would be 600×10^{-6} seconds in engineering notation. The 10^{-6} order of magnitude indicates that the units will be **micro**seconds, and the mantissa of 600 can be used directly: the time interval is 600 microseconds.

While it is generally preferred to have the digits to the left of the decimal point be in the range from 1 to 999, it can occasionally be useful to relax even this rule. If the other time intervals being discussed are in milliseconds (1 millisecond = $^{1}/_{1000}$ seconds = 10^{-3} seconds), for example, it may be better to state the number above as 0.6×10^{-3} seconds, which is easily recognized as 0.6 **milli**seconds. The usual procedure is to use the same exponent, and thus the same engineering unit, for all of any group of numbers that are being compared.

Thus the sequence is to [i] express the number in engineering notation, [ii] use the exponent in the order-of-magnitude part of the number to determine which prefix to use for the appropriate engineering unit, and [iii] express the number in that unit, *which makes it unnecessary to state the exponent*. Thus **34,600 grams** is restated as **34.6×10^{3} grams**, which is then referred to as **34.6 *kilo*grams**, or **34.6 kg**. This final form has all the advantages: short, no nonsignificant zeros, and no exponential $\times10^{n}$ term.

Conversion to Engineering Notation

Starting from ordinary notation: Use exactly the same method of moving the decimal point and adjusting the exponent as was used for scientific notation, but now take steps of three rather than single steps. Thus **23,800,000** ↔ $\mathbf{23{,}800 \times 10^3}$ ↔ $\mathbf{23.8 \times 10^6}$ (note that the commas already conveniently mark off groups of three decimal places). A small-number example is **0.0000306** ↔ $\mathbf{0.0306 \times 10^{-3}}$ ↔ $\mathbf{30.6 \times 10^{-6}}$ (some people put a space between every three numbers in a long decimal fraction, to serve the same function as commas).

Starting from scientific notation: This is even easier. If the exponent is already a multiple of 3, there is nothing more to do. If not, multiply the mantissa by 10 and decrease the exponent by 1 until a multiple of three is reached. This cannot require more than two steps, and the same method is used for large and small numbers. So a sequence needed for a number like $\mathbf{3.2{\times}10^5}$ ↔ $\mathbf{32{\times}10^4}$ ↔ $\mathbf{320{\times}10^3}$ (320 **kilo**bytes) is as difficult as it gets.

If the same number was converted to $\times10^6$ form rather than the $\times10^3$ form shown above, the sequence would be $\mathbf{3.2{\times}10^5}$ ↔ $\mathbf{0.32{\times}10^6}$ bytes, or 0.32 **mega**bytes. Whether it is preferable to express the number in kilobytes or megabytes depends on the use that will be made of it. If the number is being combined or compared with other megabyte values, then that unit would be preferable. Otherwise, it will usually be more convenient to refer to 320 kilobytes than to 0.32 megabytes, since avoiding numbers starting with zero was one of the reasons for adopting scientific and engineering notation to begin with.

Converting values from engineering units back to scientific or ordinary notation: The first step in converting a number expressed in engineering units back into one of the other forms is to put it into exponential engineering notation. Thus a value of 25 **micro**meters would be restated as $25{\times}10^{-6}$ meters. It could then either be converted to scientific notation as before $\mathbf{25{\times}10^{-6}}$ ↔ $\mathbf{2.5{\times}10^{-5}}$ (note that this will take only one or two steps of the decimal point), or the decimal point can be moved the full 6 steps to give **0.000025** in ordinary notation. A similar larger-than-10 example is 320 **kilo**meters, which would be restated as $320{\times}10^3$ meters, which is $3.2{\times}10^5$ in scientific notation and 320,000 in ordinary notation.

An example of converting to and from exponential notation is finding the total surface area of an optical fiber whose diameter is very small, 25 micrometers, but whose length is very long, 1.5 kilometers. The formula in this case is $area = 3.14 \times diameter \times length$, but it assumes (as do almost all formulas) that the diameter and length are in the same units. Square inches, square miles, or square meters are typical units of area, but what is the meaning of micrometers multiplied by kilometers, or inches multiplied by miles? The solution is to convert both diameter and length into the same units before applying the formula. Using the unit of meters (on which the stated diameter and length values are both are based) gives this result:

$$\begin{aligned} \textit{area} &= 3.14 \times \textit{diameter} \times \textit{length} \\ &= 3.14 \times 25\ \mu\text{m} \times 1.5\ \text{km} \\ &= 3.14 \times 25{\times}10^{-6}\ \text{meter} \times 1.5{\times}10^{3}\ \text{meter} \\ &= 3.14 \times 25 / 1000000 \times 1.5 \times 1000\ \ \text{meter}^{2} \\ &= 0.118\ \ \textit{or}\ \ 188{\times}10^{-3}\ \text{square meters.} \end{aligned}$$

PROBLEMS:

(1) Identify the numbers that meet all the rules for being expressed in scientific notation:

[a] 6.022×10^{23} [b] 12×10^{-3} [c] -2.9×10^{-5} [d] 0.42×10^{3} [e] 3.1416

(2) Express the following numbers in scientific notation *(any trailing zeros are* ***not*** *significant)****:***

[a] 0.00184 = ________________

[b] 1000 = ________________

[c] 5,690,000,000 = ________________

[d] 0 = ________________

[e] -271.2 = ________________

(3) Convert these scientific-notation numbers back to ordinary notation:

[a] 2.7×10^{8} = ________________

[b] -1.1×10^{-4} = ________________

[c] 7.482×10^{2} = ________________

(4) Convert the following numbers to engineering exponential notation:

[a] 1728 = ________________

[b] 1.2×10^{-7} = ________________

[c] 281,000,000 = ________________

[d] 6.022×10^{23} = ________________

[e] -0.03 = ________________

(5) Convert the following quantities to the indicated engineering units:

[a] 0.0004372 amperes = ____________ mA = ____________ μA

[b] 93,500 watts = ____________ kW = ____________ MW

[c] 0.000000178 seconds = ____________ μs = ____________ ns

[d] 2.6×10^{-4} meters = ____________ μm = ____________ mm

[e] 1.2×10^{-2} grams = ____________ mg = ____________ g

[f] 3.6×10^{7} bytes = ____________ MB = ____________ GB

(6) Complete the following:

[a] 24.5 kW = ____________ W = ____________ MW

[b] 15 seconds = ____________ ms = ____________ μs

[c] 3.25 mg = _____________________ grams

[d] 20 ns = _____________________ seconds

[e] 8 GB = _____________________ bytes

ANSWER KEY – Combining Numbers – *How accurate are the results?*

[1] A city utility company rounds off 2000 electricity bills to kilowatt-hours by truncation to whole-number values.

[a] What is the largest possible total rounding error?

[b] The smallest?

[c] The most likely?

[d] How would these answers change if the rounding were to the ***nearest*** kilowatt-hour?

[a] If each bill is rounded by truncation, the largest possible error happens when the meter reading for a bill was just below an integer value, like 123.9999 KWH. That would give a rounding error of just less than 1 KWH. In the very unlikely but not impossible event that all the bills had that much error, the total error for 2000 bills would be ***just less than 2000 KWH****.*

[b] If every meter reading was exactly on a whole-number value (such as 123.0000 KWH), the rounding error would be 0 for each case. This means that ***the smallest possible total rounding error is zero*** *KWH.*

[c] Since everyone uses many kilowatt-hours, the fractional part of the reading is just as likely to be any reading from 0 up to 1 KWH – thus the error should average ½ KWH for each meter, or ***1000 KWH for the most likely total****.*

[d] The maximum error for nearest-neighbor rounding to whole numbers is ½ rather than just less than 1, and is equally likely to be positive or negative. This means that with nearest-neighbor rounding, the maximum possible error would be ***just less than 1000 KWH*** *(in the case when all the readings are just under ½ off (always in the same direction) from a whole number, and the*

minimum error would still be zero. The average size of a nearest-neighbor error (ignoring whether it is positive or negative) would be just under ¼, but the expected value for the sum of all 2000 nearest-neighbor rounding errors is much less than ¼×2000 = 500KWH, because most of the errors would cancel each other. (To compute the expected total error accurately would require techniques that will be covered later – they predict an error of approximately 11.2 KWH, which is equally likely to be positive or negative.)

[2] A company reports assets of $24,000 and debts of $22,000, both rounded to the nearest thousand dollars. Net worth (also called "equity") is assets minus debts. What is the possible range of actual values for the net worth of this company?

*Assets could really be as high as $24,500 or as low as $23,500. Debts could really be as high as $22,500 or as low as $21,500. Combining the highest assets and the lowest debts would give **$24,500 - $21,500 = $3,000**, for the largest net worth. The worst case would come from combining the lowest possible assets and the highest possible debt: **$23,500 - $22,500 = $1,000**. Note that the possible error of the difference is bigger than that of either assets or debts.*

[3] The reported incidence of AB blood type in the United States is 4%. Combining this value with the U.S. population of 281,421,906 reported by the 2000 census, how accurately can the number of Americans with AB blood type be computed?

*Since the incidence percentage is stated to the nearest whole percent, this is apparently a rounded number representing an actual value of a low as 3.5% or as high as 4.5%. These percentages applied to the total population of 281,421,906 would give 9,849,767 for 3.5% and 12,663,986 for 4.5%. The difference between the highest and lowest possibilities is thus 12,663,986 – 9,849,767 = **2,814,219**.*

A faster way to get the same result is to multiply the total range of rounding error, 1% in this case, by the base of which a percentage is being taken: ***1% of 281,421,906 is 2,814,219****.*

[4] The ratio of debt to net worth is an often-used measure of the financial health of a company. A company with debts of exactly $125 and net worth (assets minus debts) of exactly $50 has a value of 125 / 50 = 2.5 for this ratio. Using the figures from problem [2], compute the possible range of this ratio for that company, taking into account the possible rounding errors.

The answers from [2] give a possible range from $21,500 to $22,500 for the company's actual debt and from $1,000 to $3,000 for the net worth. A ratio will be biggest when the numerator is largest and the denominator is smallest, giving ***$22,500 / $1,000 = 22.5*** *as the* ***highest*** *possible debt-to-net-worth ratio.*

Using the smallest numerator and the largest denominator gives ***$21,500 / $3,000*** *≈* ***7.2*** *as the* ***lowest*** *possible actual value consistent with the rounded numbers that were reported.*

Note that dividing by a small approximate number can result in very large variation in the result.