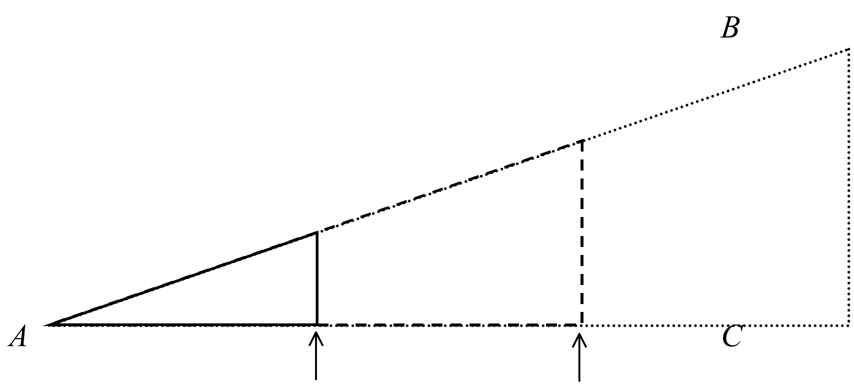
APPENDIX B-2: ANGLE SIZE

CALCULATING ANGLE SIZE IN RIGHT TRIANGLES

Review

The main conclusion of the previous lesson was that right triangles which have the same shape (that is, ones in which each angle in the first triangle is equal to an angle in the second triangle) also have the same trigonometric ratios. This is because all the sides of the second triangle are larger or smaller than the corresponding sides of the first triangle by the same proportion (called the *scaling factor* in the table below), so that the result when one side is divided by another does not change.

The figure below, in which three such similar triangles are overlaid with the corresponding angles *A* having the same corner vertex and the same orientation, illustrates this conclusion. (A similar set of triangles can be made from the cut-out triangle distributed in the previous class by folding the triangle at one or more places on the side between *A* and the right angle.)



fold points (align the paper along the AC edge)

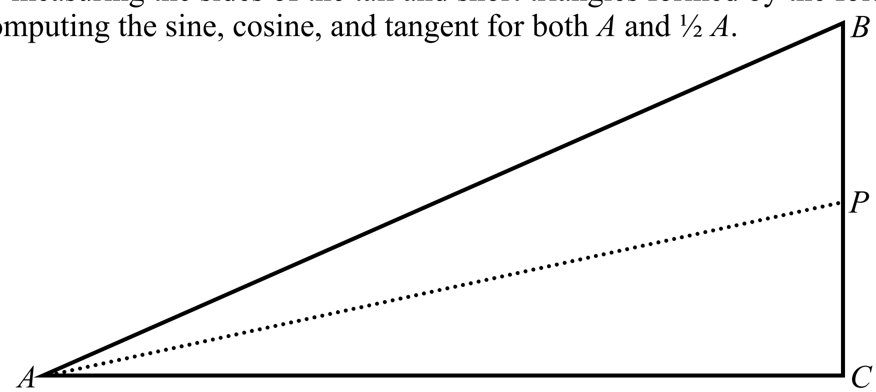
Exercise 1: Using your folded cut-out triangle (or the diagram above), measure the lengths of the sides of each of the triangles, determining the scaling factor (relative to the smallest triangle) and the main trigonometric ratios for each triangle.

	Smallest triangle	Medium triangle	Largest triangle
Length of hypotenuse			
Length of side opposite to *A*			
Length of side adjacent to *A*			
Scaling factor (compared to smallest)	1		
sin *A*			
cos *A*			
tan *A*			

Relationship of trigonometric ratios to angle size

In the right triangle shown to the right, the angle at *A* has been divided in half by a dotted line, which has been extended until it reaches the opposite side at the point *P*. This forms a "short" triangle APC with an angle half the size of the original "tall" triangle ABC. *Form a similar figure from your cut-out triangle by folding it in half along either of the corners that is not a right angle.*

What effect does cutting the angle in half have on the trigonometric ratios? Find out by measuring the sides of the tall and short triangles formed by the fold and then computing the sine, cosine, and tangent for both *A* and ½ *A*.



Exercise 2: *Comparing ratios for a half-angle*

	Hypotenuse	opposite side	adjacent side	Sine	Cosine	Tangent
Triangle A P C	AP =	PC =	AC =	$^{PC}/_{AP}$ =	$^{AC}/_{AP}$ =	$^{PC}/_{AC}$ =
Triangle A B C	AB =	BC =	AC =	$^{PC}/_{AB}$ =	$^{AC}/_{AB}$ =	$^{BC}/_{AC}$ =

Ratio for Δ APC divided by ratio for Δ ABC			

Which ratios are made smaller by cutting the angle in half? larger? Which ratio changed the most? Do any of the ratios change by a factor of exactly 2 or exactly ½? Do you see any easy pattern that would let you predict the ratio from the angle or vice-versa?

How the size of an angle is expressed

Before discussing how to relate the trigonometric ratios of an angle to its size, it will be useful to review how the size of an angle is described. Angles are usually stated in *degrees*, whose size is defined so that a right angle equals exactly 90 degrees. The other angles in a right triangle are each less than 90 degrees. In fact, these two other angles add to exactly 90 degrees (such angles are called *complementary* to each other). You can demonstrate this with your cut-out triangle by folding each side of the right angle in half, putting the points of the other angles into the corner, just filling the right angle.

All scientific calculators have a “degree” mode for expressing angles. This will usually be indicated by a small DEG or D indicator in the display. However, scientific calculators also support a “radian” mode (1 radian ≈ 57 degrees) for advanced mathematical work and a “grade” mode (1 grade = 0.9 degrees) for work in a few areas that use that angular unit. These modes, which should be avoided for this course, show indicators on the display of RAD or R and GRAD or G, respectively.

When written or printed, degree values are indicated by the ° symbol. Thus the size of a right angle is shown as 90°.

Finding the size of an angle in a right triangle

Since the trigonometric ratios for a particular angle size are always the same, and since the ratios for different-size angles in a right triangle are always different, it should be possible to determine the size of an angle from any one of its trigonometric ratios (and also to do the opposite – compute a trigonometric ratio directly from any angle size without having to draw and measure a triangle).

However, as the measurements in Exercise 2 indicate, these relationships are not obvious ones.

In the past, the trigonometric-ratio values for every small step in angle size were computed by complicated mathematical methods (sometimes taking years of work) and were then made available to users in various forms (such as tables, slide rules, and graphs) that required substantial additional training to use precisely. Now, however, the mathematical methods to convert back and forth between angle sizes and trigonometric ratios are built into inexpensive scientific calculators and values are instantly computed as needed.

Direct (angle→ratio) functions

To compute one of the standard trigonometric ratios on a scientific calculator, enter the angle size in degrees (make sure that you are in degree mode), then press the key with the abbreviation for the desired function (SIN, COS, or TAN). The ratio will be shown as a decimal fraction in the calculator display.

Thus a three-key sequence would be used to compute the tangent of a 45° angle: first the "4" key, then the "5" key, then finally the "TAN" key. The answer should be exactly 1, since in a right triangle with a 45° angle the lengths of the opposite and adjacent sides are equal because the other angle is also equal to 45°. If your answer for tan 45° is not 1, the calculator may not be in degree mode – this is an easy way to check.

Exercise 3:

Compute the indicated trigonometric ratios on a scientific calculator

sin 40° = _________ cos 20° = _________ tan 75° = __________

Inverse (ratio→angle) functions

To compute the size of an angle from one of its trigonometric ratios, you will have to press an extra key to tell the calculator that you wish to use an "inverse" trigonometric function, which converts "in reverse" from a ratio into an angle-size value. Depending on the brand of calculator, this key may be labeled "INV", "SHIFT", or "2ND", and will usually be located among the keys at the top of the keypad on the left side. In the instructions below, this will be called the "INV" key, but use whatever is right for your calculator.

First enter the ratio value, then press the INV key, then press the key that names the ratio you used. For example, the key sequence "1", "INV", "TAN" will compute the size of the angle whose tangent is exactly 1. The answer should be exactly 45, as explained above. If your answer is not 45, then either the wrong inverse-function key was used or the calculator is not in degree mode – try again until you get the correct value.

In written work, inverse trigonometric functions are designated either by putting a superscript of –1 after the function abbreviation and enclosing the ratio value in parentheses, by adding the prefix **arc-** to the function name or abbreviation, or by just putting the word "inverse" before the function name. Thus "$\mathbf{tan^{-1}(1.5)}$", "**arctan(1.5)**", and "***the inverse tangent of 1.5***" all mean the same thing. This course will most often use the **arctan**, **arcsin**, and **arcos** forms of designation, but you should be ready to understand any of these forms if they are used.

Forbidden input values

Because they ask for impossible results, some input values will give errors for certain trigonometric or inverse functions. These errors are similar to those that result from attempting to divide by zero. You should not encounter these errors are long as you work correctly with realistic problems, but they are mentioned here so that you can understand what is happening if such an error occurs due to a mistaken entry during a trigonometric calculation.

The inverse sine of 2, for example, would be the size of the angle for which the side opposite the angle would be twice as long as the hypotenuse. Since the hypotenuse of a right triangle is always longer than either of the other sides, there is no angle that produces this result. The inverse sine and inverse cosine functions are thus meaningless for ratios greater than 1, which cannot exist in real triangles.

While negative ratios or angles, and angles greater than 90°, will not arise in computations with simple right triangles, such values can be legitimate under certain circumstances, some of which will be covered later in this course. If such values are unexpected, however, they may well indicate an error and should be taken as a warning signal to check the calculation. (That is a good idea for any unexpected result.)

Computing angle size from measurements of side length

Knowing how to use inverse functions permits you to deduce angle sizes from length measurements, which are often much easier to obtain. For example, you can now determine the size of the angles in any of the right triangles whose size you have measured in earlier exercises.

Exercise 4: *Use your calculator to compute the size of the full and half angles from Exercise 2, using the ratios that you calculated there from measured values. Do the values based on different trigonometric ratio give about the same angles? Are the computed angle values close to the expected 2-to-1 ratio?*

Full angle: **(from sine)**________ **(from cosine)** ________ **(from tangent)** ________

Half angle: **(from sine)**________ **(from cosine)** ________ **(from tangent)** ________

Also check these answers by direct measurement with a *protractor*, which is used to measure angles in a way similar to how length is measured with a ruler. Protractors are marked to indicate angle size directly in degrees.

Full angle: **(from protractor)**__________ Half angle: **(from protractor)**__________

Note that computing an angle requires finding the inverse function of only a single ratio, and thus depends on knowing only two of the sides of the right triangle. This is often important in practical situations, where it may be difficult to measure one of the sides. Once the size of an angle is determined, it can be used to compute the other ratios.

Using knowledge of an angle and the length of one right-triangle side to compute another side

Even more common are situations where knowledge of the size of an angle in a right triangle can be used to directly compute a trigonometric ratio, which can then be multiplied by (or sometimes divided into) the length of another side to find the length of an unknown side you need to know.

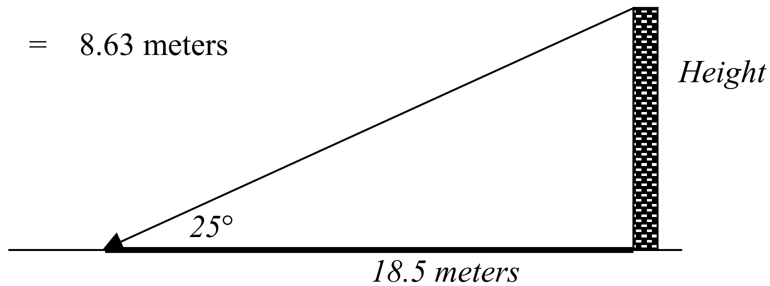
For example, if the angle of elevation of the sun is known to be 25° at 9 am and the length of the shadow cast by a tower is measured to be 18.5 meters at that time, the height of the tower can be computed as follows:

$$\frac{\text{Height of tower}}{\text{Length of shadow}} = \text{tangent of angle of elevation}$$

$$\frac{\text{Height}}{18.5 \text{ meters}} = \tan(25°) \quad º \quad 0.466$$

Height = 18.5 meters ÿ 0.466

Height = 8.63 meters



Note: *<u>How many decimal places should be retained in answers?</u>*

Calculators typically produce answers with 10 or more decimal places when a value cannot be expressed exactly in decimal notation. Thus 1/3 will be expressed as 0.333333333. When working with lengths or angles that have been mechanically measured to the nearest millimeter or degree, this gives an exaggerated idea of how accurate the results are, since if a two-digit measurement had been rounded the next value, the results would typically change by about 1%.

Later in the course we will discuss this question in detail. For now it will be sufficient if you keep three decimal places for trigonometric ratios and two decimal places for lengths or angle sizes.

HOMEWORK – CALCULATING ANGLE SIZE IN RIGHT TRIANGLES

[1] Fill in the values in the table below (retain 3 decimal places)

	10°	20°	30°	40°	50°	60°	70°	80°
Sin								
Cos								
Tan								

What relationship does this table reveal between the sine and cosine values?

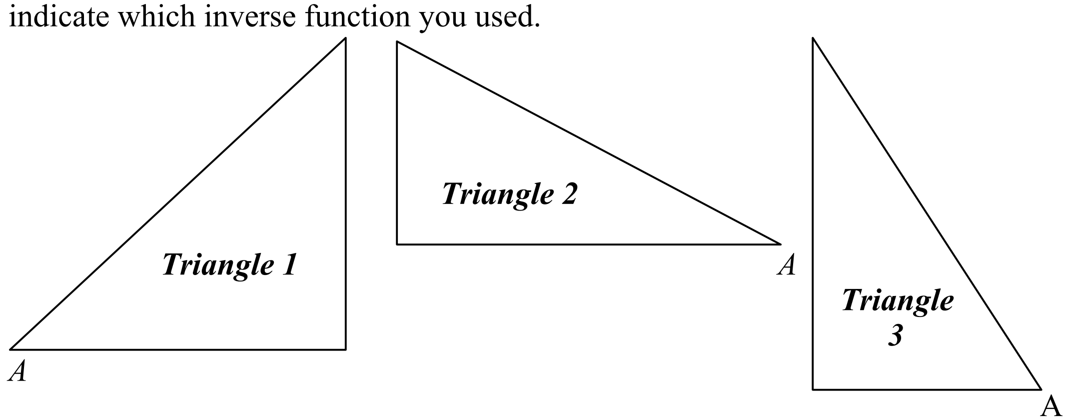
[2] Based on the values in problem 1, compute the values for $^{sin}/_{cos}$ and $^{1}/_{tan}$.

	10°	20°	30°	40°	50°	60°	70°	80°
$^{sin}/_{cos}$								
$^{1}/_{tan}$								

What relationship do these values suggest for the tangent value?

[3] Find the sine, cosine, and tangent of 89°. Also find the sine, cosine, and tangent of 1°.

[4] Applying the inverse trigonometric functions on your calculator to the ratios you calculate from the side lengths measured to the nearest millimeter, find the angle *A* for each of these triangles. In your work, show your measurements and indicate which inverse function you used.



[5] Use the same measurements as in problem 4 to find the other non-right angle in the same triangle. [Hint: the same sides can be used to give a different trigonometric ratio for that angle.]

[6] Measure both non-right angles in each triangle in problem 4 with a protractor, writing down your results. How well do these match your computed values?

[7] Test how sensitive your answers are to measurement error by this method:

[a] First, increase the numerator value in your ratio by 1 millimeter and decrease the denominator value by 1 millimeter, then recompute the ratio and calculate the angle size using these values. (This will give a *larger* ratio value than you got originally.)

[b] Then *decrease* the numerator value in your ratio by 1 millimeter and *increase* the denominator value by 1 millimeter, and recompute the ratio and calculate the angle size using these values. (This will give a *smaller* ratio value than you got originally.)

[c] Compute how much difference there is in the two ratios, compared to your original value.

[d] Compute how much difference there is in the two calculated angle sizes, compared to your original value.