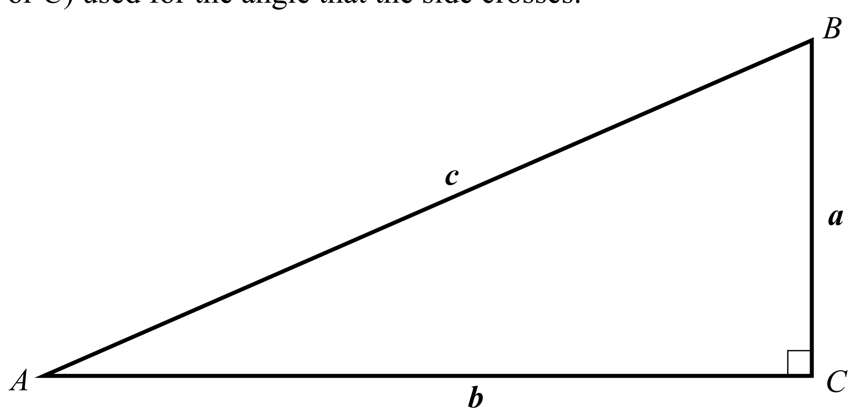
RELATIONSHIPS BETWEEN RIGHT-TRIANGLE SIDES

Trigonometric-ratio relationships

The right triangle below, in which all three angles and all three sides are labeled (C is also marked with the small square in the corner that is used to specify a right angle), will be used throughout this lesson to illustrate the relationships between the sides, angles, and trigonometric ratios of right triangles. The sides are labeled with the lower-case letter (***a***, ***b***, or ***c***) matching the uppercase letter (*A*, *B*, or *C*) used for the angle that the side crosses.



The hypotenuse of this triangle is side ***c***. The side ***a*** is opposite to angle *A* and is adjacent to angle B. The side ***b*** is opposite to angle *B* and adjacent to angle *A*. Using the definitions of the trigonometric ratios, we can see that sin A and cos B are the same ratio:

$$\sin A = \frac{\text{side opposite to } A}{\text{hypotenuse}} = \frac{a}{c} = \frac{\text{side adjacent to } B}{\text{hypotenuse}} = \cos B$$

In the same way, we can show that cos *A* is the same ratio as sin *B*.

$$\cos A = \frac{\text{side adjacent to } A}{\text{hypotenuse}} = \frac{b}{c} = \frac{\text{side opposite to } B}{\text{hypotenuse}} = \sin B$$

Since A and B are *complementary* to each other (that is, they add up to 90°), the above result can be expressed as either of these equations:

sin(angle) = cos(90°- angle)   or   cos(angle) = sin(90°- angle)

Exercise 1: *Verify that both the equations above are true for some arbitrary particular angle of your choice between 1 and 89 degrees.*

angle = ____°   sin(angle) = _________   cos(90°- angle) = _________

cos(angle) = _________   sin(90°- angle) = _________

The tangent ratios for A and B are both formed from the lengths of the sides ***a*** and ***b***, but they change places in the numerator and denominator of the ratio. Thus the values for tan *A* and tan *B* are reciprocals of each other.

$$\tan A = \frac{\text{side opposite to } A}{\text{side adjacent to } A} = \frac{\boldsymbol{a}}{\boldsymbol{b}}$$

$$\tan B = \frac{\text{side opposite to } B}{\text{side adjacent to } B} = \frac{\boldsymbol{b}}{\boldsymbol{a}}$$

A further useful relationship is that the tangent of an angle is equal to the ratio of the sine of that angle to the cosine of the same angle. This can be shown from the definitions:

$$\frac{\sin A}{\cos A} = \frac{\boldsymbol{a}/\boldsymbol{c}}{\boldsymbol{b}/\boldsymbol{c}} = \frac{\boldsymbol{a}}{\boldsymbol{c}} \times \frac{\boldsymbol{c}}{\boldsymbol{b}} = \frac{\boldsymbol{a}}{\boldsymbol{b}} = \tan A$$

Exercise 2: *Verify that the tangent of some particular angle is equal to the ratio of the sine of that angle to the cosine of the angle.*

angle = °	sine =	cosine =	sin/cos =	tan =

The surprising relationship

The relationships between trigonometric ratios that have been shown so far follow from the definitions of the ratios in a straightforward way. The proofs are shown and are easy to follow. However, the most important relationship of this kind is not nearly as obvious. It is indicated by the table below, which gives the sines and cosines of various angles, then lists the "square" of these values (that is, the value multiplied by itself) for each angle. The surprise is the value of the sum of the squares, $(\sin A)^2 + (\cos A)^2$, which is traditionally written as $\sin^2 A + \cos^2 A$ in mathematical work.

Angle A	10°	20°	30°	40°	50°	60°	70°	80°
sin A	0.17365	0.34202	0.50000	0.64279	0.76604	0.86603	0.93969	0.98481
cosine A	0.98481	0.93969	0.86603	0.76604	0.64279	0.50000	0.34202	0.17365
$(\sin A)^2$	0.03015	0.11698	0.25000	0.41318	0.58682	0.75000	0.88302	0.96985
$(\cos A)^2$	0.96985	0.88302	0.75000	0.58682	0.41318	0.25000	0.11698	0.03015
$\sin^2 A + \cos^2 A$								

Exercise 3: *Compute the sum* $\sin^2 A + \cos^2 A$ *for 10°, 20°, 30°, and 40°.*

It can be shown that this $\sin^2 A + \cos^2 A = 1$ equation is exactly true for all angles (although a proof takes more than checking a few values with a calculator). This "Pythagorean Identity" is an important and useful mathematical fact in itself, but it can be transformed into an even more useful result by expressing it in terms of the lengths of the sides of a right triangle which has A as one if its angles. Using the labeled right triangle introduced at the beginning of this lesson, where we saw that

$\sin A = {}^{a}/_{c}$ and $\sin B = {}^{b}/_{c}$

we can see that the equation

$$\sin^2 A + \cos^2 A = 1$$

can also be stated as

$$({}^{a}/_{c})^2 + ({}^{b}/_{c})^2 = 1$$

which in turn can be stated as

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$$

which is equivalent to

$$\frac{a^2 + b^2}{} = 1$$

c^2

If both sides of this equation are multiplied by the denominator term c^2, the equation

$$a^2 + b^2 = c^2$$

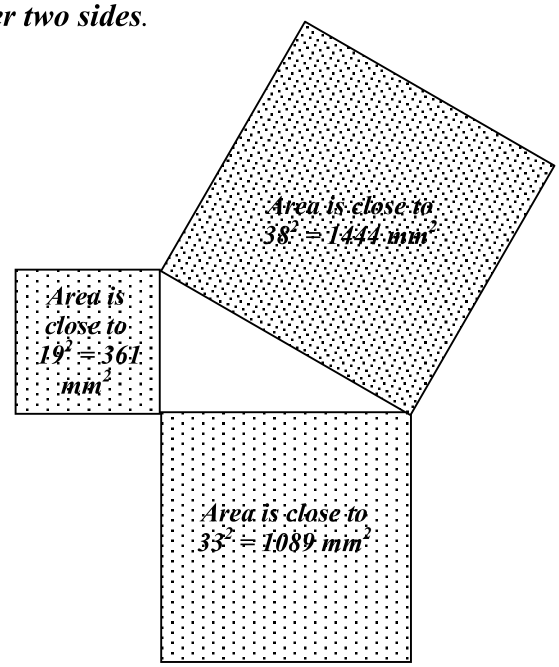
is obtained, which can be expressed in words as

The square of the length of the hypotenuse of a right triangle equals the sum of the squares of the lengths of the other two sides.

This is called the *Pythagorean Theorem*. When it was first proved by ancient Greeks, it was expressed in a geometrical form that uses the term "square" literally, not algebraically:

The area of ***the square on the hypotenuse*** *of a right triangle is equal to the sum of the areas of the* ***squares on the other two sides****.*

The figure is a right triangle whose sides measure 19, 33, and 38 mm (to the nearest millimeter). The square of 38 is 99.6% of sum of the squares of 19 and 33, which is as close to an exact match as can be expected when working with measurements of that accuracy. The small errors from rounding that measured values must have is one reason that mathematical proofs depend on logic, not on measurement.



Using the Pythagorean Theorem

In many practical situations, two of the three sides of a right triangle can be determined by measurement or by the conditions of the problem. The advantage of the Pythagorean Theorem is that in such cases the third side can be determined without having to determine the angles of the triangle or to calculate any trigonometric ratios. (This was particularly important before calculators made such calculations easy.)

Since the theorem is about the *squares* of the lengths, however, rather than the lengths themselves, it is usually necessary to find a *square root* at the end of the calculation. (The square root of a number is the value that will give the number as a result when squared – for example, 3 is the square root of 9.) All calculators have a key to find square roots; it is usually marked with the mathematical “radical” symbol $\sqrt{}$.

Here is an example problem:

What is the diagonal distance across a rectangular lot that is 23 meters long and 45 meters deep?

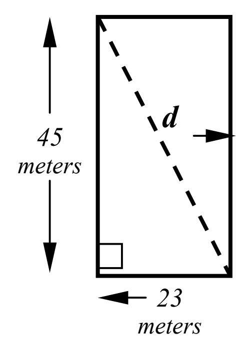
Since the field is rectangular, its width and depth are the sides of a right triangle whose hypotenuse is the diagonal distance ***d*** that we wish to find. Thus

$$\boldsymbol{d}^2 = 23^2 + 45^2 = 529 + 2025 = 2554 \text{ meters}^2$$

Taking the square root of both sides of this equation gives

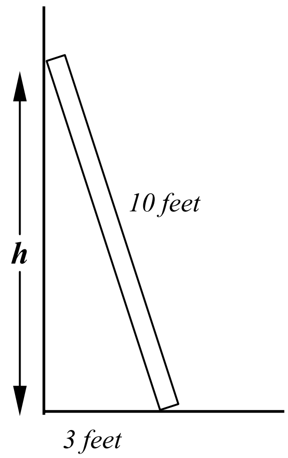
$$\boldsymbol{d} = \sqrt{2554} \cong 50.537 \text{ meters}$$

(As is usual with square roots, this value is a rounded approximation, since no value with this precision gives exactly 2554 when squared.)



Here is a different example, where a side is unknown rather than the hypotenuse:

If the lengths of the hypotenuse and of one side are known, finding the length of the other side can be done by subtracting the square of the length of the known side from the square of the length of the hypotenuse. For example, if a 10-foot ladder is placed with its base 3 feet from a vertical wall, how high on the wall will the top reach?



Let us call the desired height $\boldsymbol{h}$.

The Pythagorean Theorem tells us that

$$h^2 + 3^2 = 10^2$$

This is the same as

$$h^2 = 10^2 - 3^2 = 100 - 9 = 91$$

so that

$$h = \sqrt{91} \cong 9.54 \text{ feet}$$

Limitations of the Pythagorean Theorem

Remember that the $a^2 + b^2 = c^2$ equation ***only applies to right triangles***. It cannot be used to find the standoff of a ladder placed against a leaning wall, or the diagonal across a non-rectangular field. *Before applying the Pythagorean Theorem to a problem, you must make sure that the sides whose lengths you are using in the equation form a right triangle.*

Later in the course we will discuss how to use a more powerful (but somewhat more complicated) method that works with all triangles, whether or not they contain a right angle. The Pythagorean Theorem is a special case of this more general "Law of Cosines".

Using the Pythagorean Theorem to make a right triangle

Not all true statements can be "turned around": for example, all triangles are formed from straight lines, but it is ***not*** true that all figures formed from straight lines are triangles.

But the Pythagorean Theorem works both ways. If the sum of the squares of the lengths of two of the sides of a triangle equals the square of the length of the third side, then the angle opposite the third side is a right angle.

Because $3^2 + 4^2 = 5^2$ (that is, $9 + 16 = 25$), a triangle whose sides have the lengths 3, 4, and 5 will thus have a right angle opposite the side whose length is 5. This is also be true of any triangle whose sides are in this same proportion, such as 6, 8, and 10 or 30, 40, and 50. This fact was used by the ancient Egyptians in making right triangles for use in construction and surveying. Although the set {3,4,5} contains the smallest-number case, there are infinitely many other “Pythagorean triplets” of whole numbers. However, most right triangles will not have whole-number side lengths.

To test the size of an angle of a triangle, compare the square of the length of the side opposite to it to the sum of the squares of the lengths of the other two sides.

If the square of the length equals the sum, the angle is a right angle.

If it is less than the sum, the angle is an *acute* angle (less than 90°)

If it is greater than the sum, the angle is an *obtuse* angle (greater than 90°)

Exercise 5:

Classify (as right, acute, or obtuse) the largest angle in each listed triangle.

Triangle 1 has sides with lengths 11.2, 6.3, and 8.4 right__ acute__ obtuse__

Triangle 2 has sides with lengths 1.2, 0.5, and 1.3 right__ acute__ obtuse__

Triangle 3 has sides with lengths 5.6, 3.2, and 4.4 right__ acute__ obtuse__

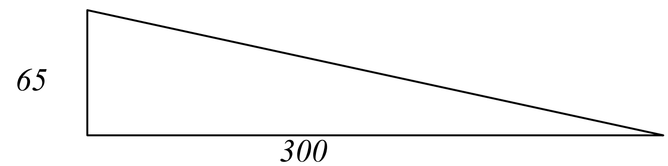
Triangle 4 has sides with lengths 7.2, 6.3, and 8.1 right__ acute__ obtuse__

HOMEWORK – RELATIONSHIPS BETWEEN RIGHT-ANGLE SIDES

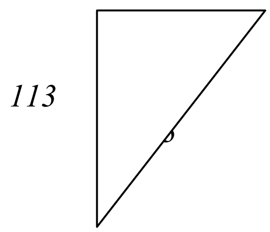
Work all problems on a separate sheet of paper, showing your work in full.

[1] Find the length of the side whose length is not indicated.

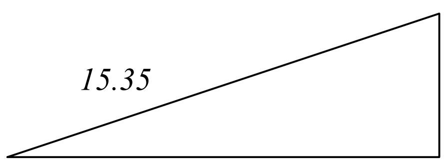
[a]



[b]



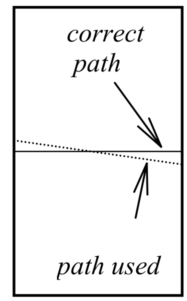
[c]



[2] How long is the diagonal of a rectangle whose width is 17.203 meters and whose length is 12.341 meters?

[3] If the sine of an angle is 0.582, compute the cosine of that same angle by two different methods. [Hint: Use inverse trigonometric functions during one method, and find a square root during the other.]

[4] Rather than measuring straight across a board whose actual width is exactly 12 inches, a workman accidentally measures so that one end of his ruler is offset from the point straight across the board by 1 inch (see illustration to the right).

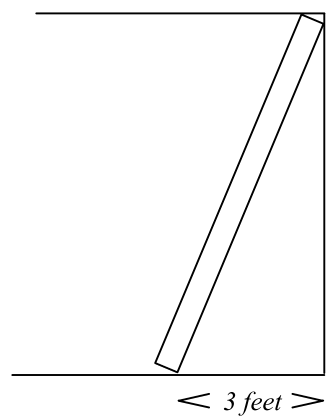


[a] How much longer will his measurement be than it should be?

[b] What percentage is this error of the board's width?

[5] If one location is 45 kilometers south and 15 kilometers west of another, how far apart are they in a straight line (ignoring the curvature of the earth)?

[6] A 9-foot ladder is placed so that it touches the ceiling where it meets the wall. The base of the ladder is found to be 3 feet from the wall. (See illustration to the right.)



[a] How high is the ceiling? (To make working this part of the problem easier, assume that the ladder has zero thickness.)

[b] *Advanced version of the problem (but we have covered all the tools you need to solve it):* If you take into account the fact that the ladder is 6 inches thick, how high would you calculate that the ceiling is?

[7] The sides of a triangle are measured to be 6.5 meters, 7.2 meters, and 9.7 meters. Are these measurements consistent with a right triangle?

[8] If two sides of a right triangle are equal in length, what fraction of the length of the hypotenuse is one of the sides?