APPENDIX B-4: AREA AND PROOF

AREA – Rectangles, Triangles, and Squares

The *area* of an object is the amount of surface it covers. While it is usually more complicated to measure or compute an object's area than the lengths of its parts, the areas of figures with straight-line sides can be found by simple methods based on rectangles and triangles. These straightforward area-computation methods can also be used to supply a mathematical proof (rather than just measure-and-compute verifications) of the Pythagorean Theorem.

Area of squares

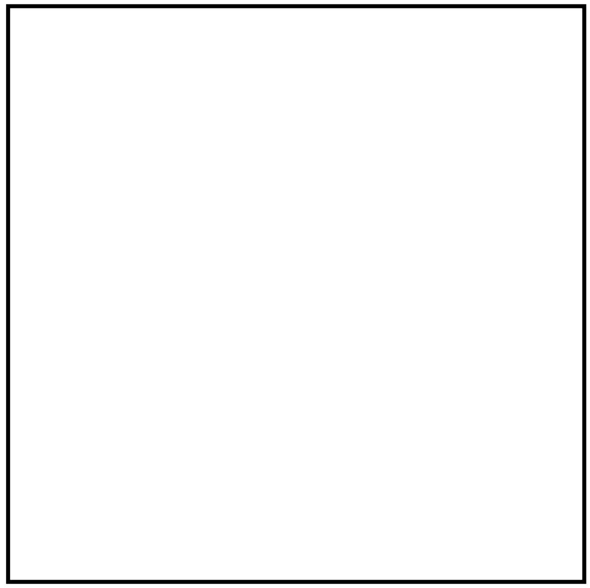


***Figure 1**. The area of a 3-inch square is 9 in^2.*

On flat surfaces, ***the area of a square is the length of the side multiplied by itself***. Thus the area of the square shown in Figure 1 is 3 inches × 3 inches = 9 square inches (another way of writing this is 9 in^2). The dotted interior lines that divide the square into a set of 9 smaller 1-inch-wide squares show why this definition of the area of a square is seen as natural.

This definition is so traditional – geometry is much older than algebra – that the terms "squaring" and "square of a number" are used throughout mathematics to refer to multiplying a number by itself, even when no reference to lengths, square figures, or surface area is intended.

area = sidelength × sidelength = sidelength^2 [formula for the area of a square]

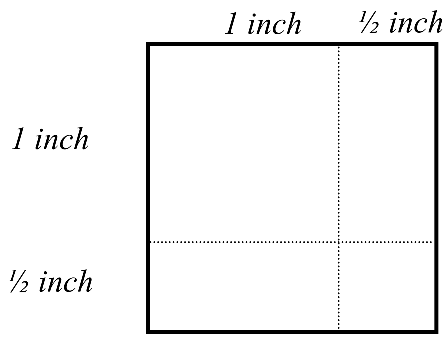


***Figure 2**. Computing the area of a square when the length of its side has a fractional part*

The same definition of area is used even if the length of the square's side is not a whole number (in whatever length units are being used). The square (Figure 2) whose side is 1.5 inches thus has an area of 1.5 inches × 1.5 inches = 2.25 in^2. This result can be illustrated by adding the full and partial squares in the figure, giving a sum of *one full-square, plus two half-squares, plus one quarter-square*:

$$\text{Total area } (in^2) = 1 + (2 \times \tfrac{1}{2}) + \tfrac{1}{4} = 1 + 1 + \tfrac{1}{4} = 2\tfrac{1}{4} = 2.25$$

Adding the areas of the pieces that the square has been divided into thus gives the same result as 1.5 inches × 1.5 inches = 2.25 in^2

Exercise 1: *Relation of a square's area to its side length*

[a] What is the area of a square whose side is 5 inches?

[b] What is the area of a square whose side is 27.2 millimeters?

[c] What is the area of a 10-inch square? A 20-inch square? In general, how much does the area of a square change when its side length is doubled?

[d] What is the side length for a square whose area is 100 mm^2 (square millimeters)? A square whose area is 200 mm^2 ? In general, how much longer does a square's side become when its area is doubled?

Area of rectangles

For rectangles, which have two sets of sides that can have different lengths, these same ideas lead to this definition: ***the area of a rectangle is the product of the lengths of two adjacent sides***. In the common case where the length of two of the sides is called the *width* and the length of the other two is called the *height*, this definition can be expressed as:

area = width × height [formula for the area of a rectangle]

But note that the definition does not depend on the orientation of the rectangle, or on the labels used for the sides, so that all the rectangles shown below have the same area. It is the length of the sides that matters, not the overall horizontal and vertical extent of the figures:

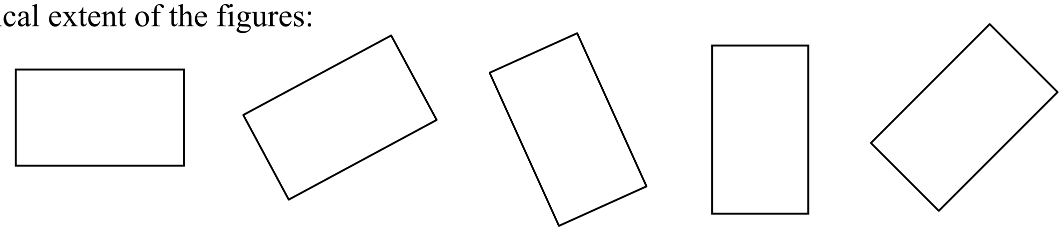


Figure 3. *Orientation doesn't affect area.*

It is also possible for rectangles with the same area to have different shapes (this is not true of squares). This is shown by the set of rectangular shapes below, each of which has an area of 900 mm^2:

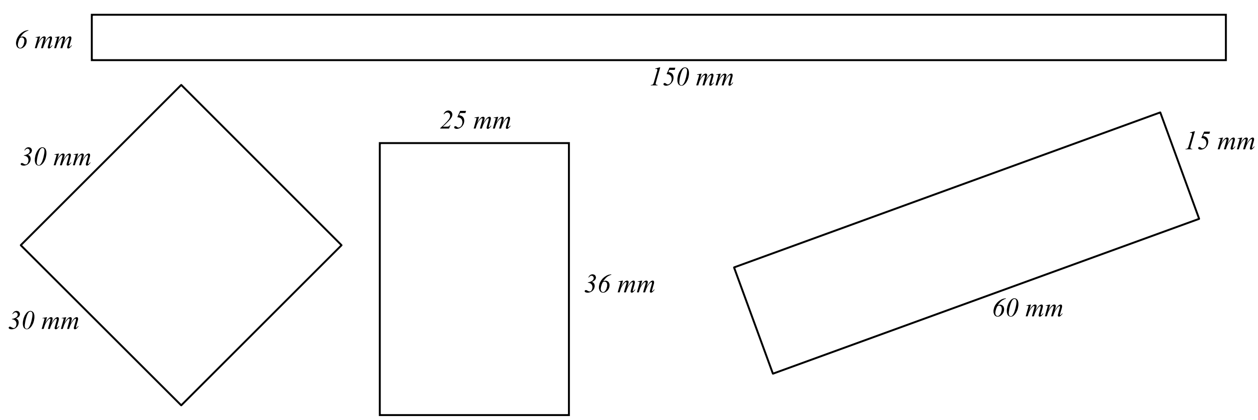


Figure 4. *Rectangles that have different shapes but have equal areas.*

Exercise 2: *Relation of the sides of a rectangle to its area*

[a] What is the area of a rectangle whose width is 3.8 inches and height is 6.5 inches?

[b] Measure one of the rectangles in Figure 3 (to the nearest millimeter) and compute its area in square millimeters.

[c] If a rectangle that has a width of 25 millimeters also has an area of 850 mm^2, what is its height?

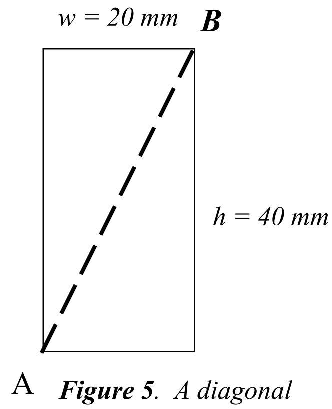


***Figure 5**. A diagonal divides a rectangle into equal halves, each of which is a right triangle.*

Area of right triangles

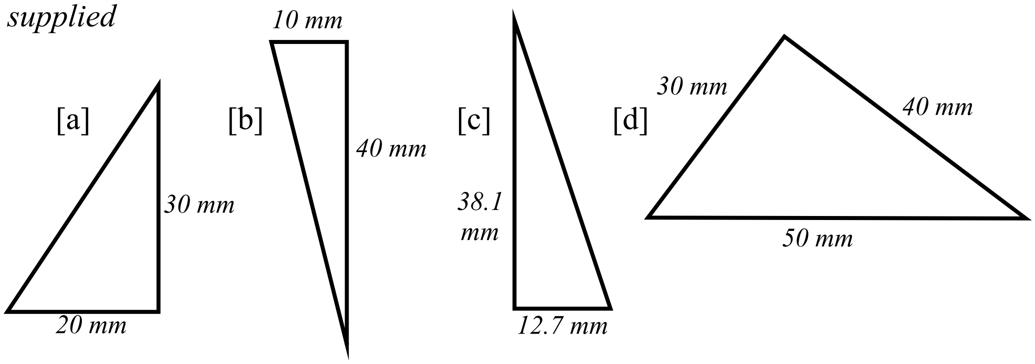
Figure 5, a rectangle whose height is 40 mm and width is 20 mm, indicates how the area of a right triangle can be computed from the area (which is 20 mm × 40 mm = 800 mm^2 in this case) of a related rectangle. The diagonal from ***A*** to ***B*** divides the rectangle into two "congruent" right triangles that have the same size and shape (that is, they would overlap exactly if placed on top of each other), which means that each of these right triangles must have an area of 400 mm^2, which is half of the area of the original rectangle.

While we used particular numbers as examples for the width and height of this triangle, the argument that the diagonal divided the rectangle into two equal triangles did not depend on what the values were, and applies just as well to all values. Thus

for a right triangle, ***area = ½ × width × height***

Note that in this formula the terms "width" and "height" refer to the sides that form the right angle, not to the hypotenuse (which in Figure 5 is the diagonal).

Exercise 3: *For each triangle, compute the area, using the measurements supplied*



Calculating the area of *ANY* triangle

Up to this point in the course, we have dealt only with a special kind of triangle: those in which one of the angles is 90°. Many results, such as the Pythagorean Theorem and the definitions of the trigonometric ratios, apply only to such "right" triangles.

But we are now ready to extend our attention to other triangles as well. As the diagrams below indicate, *it is always possible to divide a triangle into two right triangles* with an "altitude" line segment (dotted in the illustrations below) that goes by the shortest path from the vertex of the largest angle to meet the level of the opposite side.

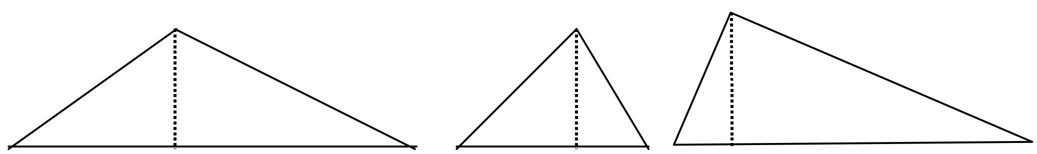


Figure 6. *An altitude to the longest side splits a triangle into 2 right triangles.*

In a triangle divided in this way, it would be possible to use the formula for right-triangle area derived earlier to find the area of each sub-triangle, then add the two areas to give the total area. But Figure 7 below shows a simpler way to compute the overall area. The wide dashed-line rectangle shown, with one pair of sides equal in length to the base of the triangle and the other pair of sides equal in length to the altitude, makes it clear that the area of the triangle is half of the area of the rectangle (each subdivided triangle is half the area of the corresponding subdivision of the wide rectangle). But the area of the wide rectangle is equal to its width (the length of the base side of the triangle) multiplied times its height (the length of the altitude to that base side).

Therefore for *any* triangle, ***area*** = ***½*** × ***base*** × ***height***

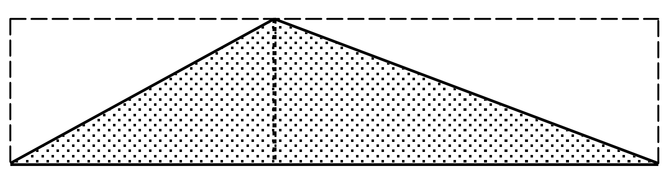
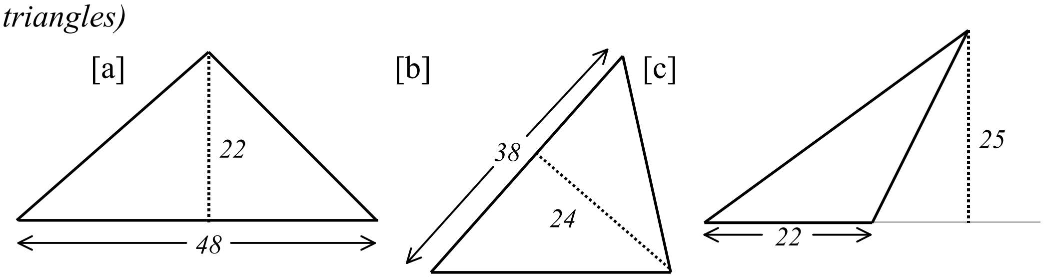


Figure 7. *The area of a triangle is half the product of the lengths of a side and of the altitude to that side.*

The same area will result regardless of which side of the triangle is chosen as the base, as long as the corresponding altitude is also used, but choosing the longest side as the base ensures that the altitude stays inside the triangle, making its measurement easier.

*Exercise 4: Calculate the area of each triangle shown (these are **not** right triangles)*



– all numbers shown are lengths in millimeters –

Mathematical proof

The derivation above of the ***area* = ½ × *base* × *height*** formula for triangle area is a simple example of a mathematical proof. The derivation was not based on measurements (which are always subject to some error) or on particular values (which might be special cases). Instead, we started with a basic idea (the ***area* = *width* × *height*** formula for the area of a rectangle) that we had good reason to believe, and then used a logical argument (that the diagonal of a rectangle always splits it into two equal halves) to produce a new conclusion related to right triangles. We then extended that conclusion to all triangles by showing how any triangle could be split into two right triangles, and showing that a wide rectangle including both right triangles can be made the basis for a general triangle-area formula based on a side of the triangle and the altitude to that side.

In a more elaborate proof, we would explain our reasoning in more detail, for example giving further arguments why the two triangles are the same size and shape (because in a rectangle each pair of opposite sides have equal length, and triangles whose sides match exactly in length have the same area). It often can be valuable to look at the reasoning of an argument that produces a formula in more detail, and in particular to be clear about what basic ideas the derivation assumed to be true, so that we can see if the formula applies to the situation where we are considering using it.

Proofs need not be elaborate to be useful. Since no one is in a position to prove all their assumptions (what would such a proof start with?), logic is always just a way of showing further consequences to what we already believe to be true. Skill at logical deduction makes it possible to dependably apply a relatively short list of conclusions that we or other people have already worked out (the most useful math results are centuries old) to a great variety of applications.

But it *is* important to keep in mind the assumptions on which a formula is based, even if the formula is accepted as true without remembering the proof. For example, the use of these area formulas requires that the surfaces on which the rectangles and triangles are drawn must be capable of being flattened out without stretching. If not, these area formulas won't work (because the smaller regions we got when we subdivided Figure 1 would not turn out to be equal squares). Thus these formulas will work for figures drawn on a plane surface like a tabletop, but not for figures drawn on a spherical ball. (Some partially-curved surfaces can be flattened, as is shown by the fact that cylinders and cones can be formed, without stretching, from a flat sheet of paper.)

A proof of the Pythagorean Theorem *(to think about, not to memorize)*

Although this course will concentrate much more on the logical application of mathematical ideas than on mathematical proof, the results already established in today's lesson are sufficient to build a proof of the surprising $a^2 + b^2 = c^2$ result about right triangles that was discovered (but not proven) in the third lesson.

The first step of the proof is to divide an appropriate larger square so that it contains both a square of side length ***a*** and a square of side length ***b***. The diagram in Figure 8 (on the next page) shows how this can be done with a square whose side has length ***a***+***b***. (Why use exactly that length? Because it makes things work just right later in the proof – proofs like this are contrived after the answer has been figured out.)

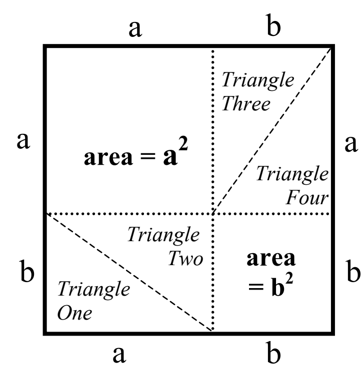


Figure 8. *A division of the square (a+b)² into 2 squares and 4 equal right triangles*

Divided as shown, the area of the square whose side length is ***a***+***b*** can be expressed as the sum of the areas of two squares (one square with side ***a*** and another square with side ***b***), plus the areas of two rectangles (each of which has one side of length ***a*** and an adjacent side of length ***b***).

Now note that the two rectangles can each be divided into two right triangles whose side lengths are ***a*** and ***b*** (giving a total of four such triangles, all of which are the same size, shape, and area). This is indicated by the diagonal dashed lines that split the rectangles.

Thus the division of this $(a+b)^2$ square is into these pieces:

– one square with side ***a*** (whose area is a^2),
– another square with side ***b*** (whose area is b^2), and
– four equal right triangles each having a side of length ***a*** and a side of length ***b***.

The proof then proceeds to divide the same ***(a+b)***^{2} square into a different pattern, as shown in Figure 9. Here, each side of the large square is still divided into two pieces, one of length ***a*** and another of length ***b***, but the dividing points are arranged and connected to form four triangles cut off from the corners of the big square, and to make an interior four-sided figure.

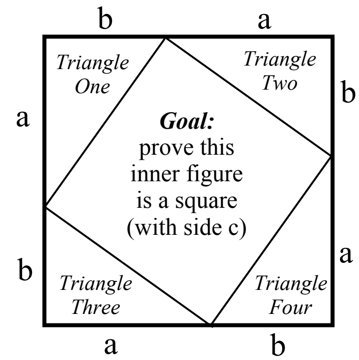


Figure 9. *An alternative division of the square $(a+b)^2$ into 1 square and 4 equal right triangles*

The next step is to show that the four-sided interior figure is a square (just because this case looks square doesn't prove anything for sure, although it encourages the use of this approach). The interior figure can be proved square by showing that two things are both true: all its sides have equal lengths, and all its angles are the same size.

The corner triangles are all right triangles (because each includes an angle from a corner of the square, all of which are right angles). Since each of these right triangles has one side of length ***a*** and another side of length ***b***, they all have the same size and shape, which implies two things that are important for this proof:

[i] All the hypotenuses of these triangles are the same length (which we will call ***c***).

[ii] The four angles labeled *A* in Figure 10 (each of which has the tangent $^a/_b$) are the same size for all of the corner triangles, as are the angles labeled *B* (each of which equals 90°-*A*).

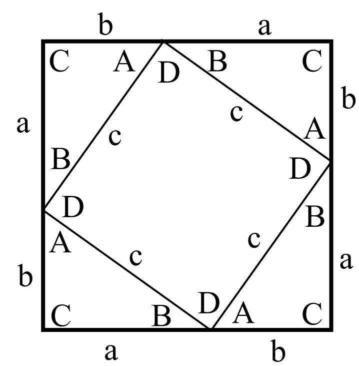


Figure 10. *The same division of $(a+b)^2$, with the other corresponding equal parts also labeled with the identical letters*

Since each angle of the inner four-sided figure is formed from a straight line from which one angle of size *A* and one angle of size *B* has been taken away, these angles are all the same size, shown as *D* in Figure 10.

*Because **the inner 4-sided figure** both:*

[i] has sides equal to each other and also

*[ii] has angles equal to each other, it **is a square**.*

The area of this square is c^2, the side length multiplied by itself.

Therefore this version of the $(a+b)^2$ square has been divided into the following pieces:

– one square with side c (whose area is c^2), and

– four equal right triangles each having a side of length a and a side of length b.

The final stage of the proof compares these two different patterns of dividing up the area of a square that has sides of length $a+b$. The key fact is that ***the four right triangles are present in the same size, shape, and number in both patterns of division***. (The locations are different, but location and orientation don't affect the area.)

In the first case (figure 8), the two squares with areas a^2 and b^2 are what is left when these four triangles are taken away. In the second case (Figures 9 and 10), the single square with area c^2 remains when the triangles are removed.

Since in both cases equal amounts of triangle area are taken away from squares that start with the same area, the remaining areas must still be equal. This proves what we set out to prove:

$$c^2 = a^2 + b^2,$$ which in words is

The square of the length of the hypotenuse of a right triangle equals the sum of the squares of the lengths of its other two sides.

HOMEWORK – AREA & PROOF

Work all problems on a separate sheet of paper, showing your work in full.

[1] What is the area of a square whose side is 6.4 inches?

[2] How long is the side of a square whose area is 3600 square feet?

[3] Measure the rectangle shown and calculate its area in square millimeters.

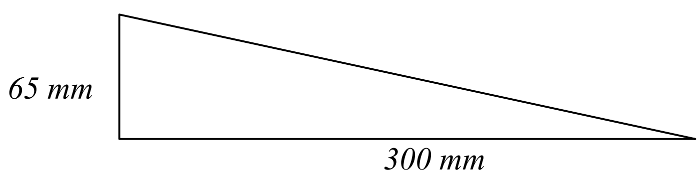
[4] A rectangular field that is 150 feet deep has an area of 6825 square feet.

[a] How wide is this field?

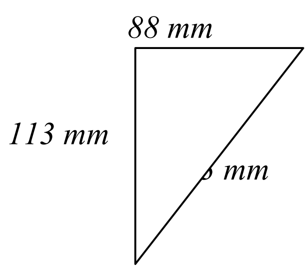
[b] How long is the diagonal across the field?

[5] Use the lengths indicated as needed to calculate the areas of these right triangles (in some cases, you may need to first calculate a length that is not directly provided).

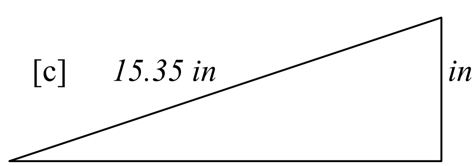
[a]



[b]

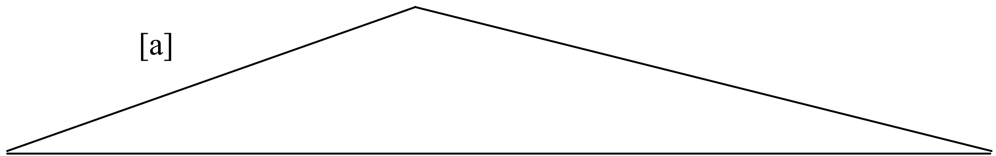


[c]

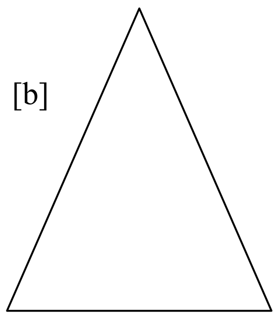


[6] Determine the area (in square millimeters) of these triangles, which are ***not*** right triangles, by making appropriate length measurements and then using those measurements to calculate the area with the appropriate formula.

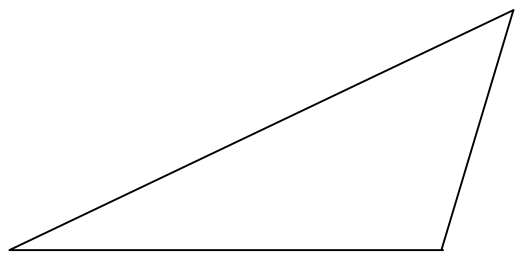
[a]



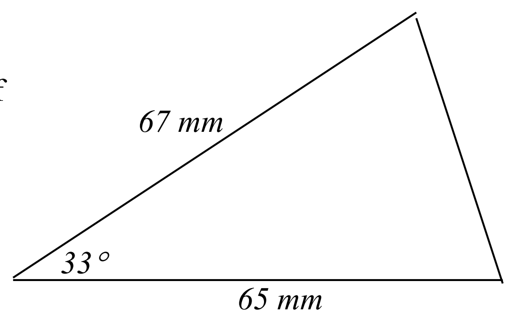
[b]



[c]



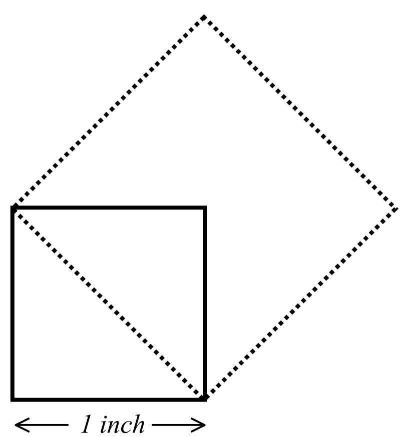
[7] In the triangle shown below (which is not a right triangle), the lengths of two sides and the size of the angle between them are known, as indicated. What is the area of this triangle?



[8] The figure to the right shows a one-inch square in which a second square (shown with dotted lines) has been constructed whose side is the diagonal of the original square.

[a] What is the area of the larger square?

[b] Can you give a proof (not depending on measurement) for your answer?



ANSWERS TO SELECTED EXERCISES:

[2] Since the area of a square can be computed by multiplying the side length times itself (that is, by *squaring* the length), we should apply the inverse process (that is, the *square root*) to the area in order to find the length.

$$Length \quad = \quad \sqrt{Area} \quad = \quad \sqrt{3600\,ft^2} \quad = \quad 60\,ft$$

[4 a] Since the area of a rectangle is the product of its two dimensions, either dimension can be computed by dividing the area by the other dimension.

$$Area \quad = \quad Width \;\mu\; Depth$$

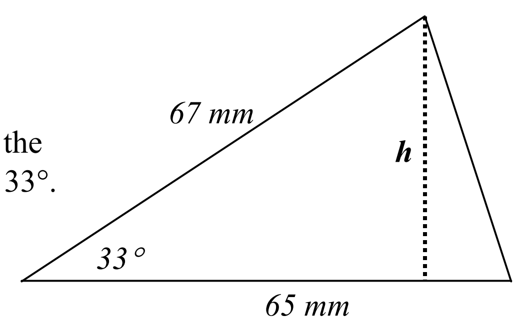
$$\text{Therefore} \qquad Width \quad = \quad \frac{Area}{Depth} \quad = \quad \frac{6825\,ft^2}{150\,ft} \quad = \quad 45.5\,ft$$

[4 b] The diagonal is the hypotenuse of a right triangle whose other sides are the sides of the rectangle, so by the Pythagorean Theorem its length can be calculated as

$$diagonal \quad = \quad \sqrt{150^2 + 45.5^2} \quad @ \quad 156.7\,ft$$

[7] In order to compute the area of a general triangle, you need to know both the length of a side and the length of an altitude to that side. In this case, we do not have the length of an altitude, so we will need to compute it from the information given.

If the lower side is taken as the base of the triangle, we can compute the height ***h*** of the needed altitude (shown as a dotted line) from the fact that the ratio of ***h*** to 67 mm is the sine of 33°.



$$\sin(33°) \quad = \quad \frac{h}{67mm}$$

Therefore $\boldsymbol{h}$ = sin(33°) ä 67 mm = 0.5446 ä 67 mm @ 36.49 mm

Using the formula for the area of a triangle,

area = ½ ä ***height*** ä ***base*** = ½ ä 36.49 mm ä 65 mm @ 1186 mm^2