APPENDIX C-3: ERROR PROPAGATION

MEASUREMENT-ERROR SENSITIVITY OF CALCULATION PROCESSES

We have already seen how it is possible to assess the quality of the measurements produced by a process: by calibration (for systematic errors), and by noting the variation in repeated measurements (for random errors). The question examined in this lesson is ***what happens with the measurement error when measurements are combined***? Do the errors cancel? Become larger? Can the effects of different combinations be predicted?

The most straightforward way to examine these questions is to do a *perturbation analysis* (this is a fancy term for “jiggle it and see what happens”). You have already done these is several cases, in problems where you were told to slightly change one of your measurements and see the effect on the result of a calculation that used that measurement. We are now going to look at this method systematically to show its implications for several types of error: offset errors, scaling errors, and random errors.

Overview of perturbation analysis for calculations using one measurement:

[1] ***Select the reference measurement value***. The reference should be typical of the value(s) that will arise in actual practice. If these vary widely, you will need to repeat the analysis with different references at each end of the range of possibilities, just as you would in a calibration process. *[Example: 50 mm as the measured side of a square.]*

[2] ***Calculate the unperturbed result***, using the reference value. Thus if you are computing the area of a square from a side-length measurement, multiply it times itself to give an area that will be the reference result value. *[Example:* $(50\ mm)^2 = 2500\ mm^2$*]*

[3] ***Choose a perturbation size***. This should be small enough that it is a plausible value for a random measurement error. A typical perturbation would be 1 mm for a ruler measurement or 1 degree for a protractor measurement, but any sufficiently small value will work, and it is often convenient to use the typical error for the measurement process if that has been determined. The exact perturbation size doesn’t matter much because we will normalize for it in step 5. *[Example: 1 mm is a reasonable perturbation for a 50-mm measurement, but if in doubt choose a smaller perturbation – the smaller the better.]*

[4] ***Calculate high and low perturbed results***. First add the perturbation to the reference measurement and then use the same calculation method as before to produce a result, then do the same thing after subtracting the perturbation from the reference. Do not round off either result. *[Example:* $(51\ mm)^2 = 2601\ mm^2$*,* $(49\ mm)^2 = 2401\ mm^2$*]*

[5] ***Compute and state the error sensitivity*** by dividing the difference in the high and low results from the difference in perturbations (twice the perturbation size with this method), then state the effect per unit of perturbation. Include a statement of the reference value on which the analysis was based (unless it doesn't matter). *[Example: (2601 – 2401) / 2 = 100, so the perturbation effect is "an error sensitivity of 100* mm^2 *of area per millimeter of error in side length for a reference side length of 50 mm"]* An alternative form of statement, appropriate in some cases, states the error sensitivity relative to the reference result. *Example: "The area of a square has an error sensitivity of 4% per millimeter of error in the side length, for a reference side length of 50 mm."*

Example 1: *Using a perturbation analysis, estimate the error sensitivity of the sine function to errors in angle measurement, using a reference value of 10°.*

The reference value, 10°, is supplied in the statement of the problem.

Applying the function to be analysed gives a reference result of **sin(10°) = 0.1736**

1° is a reasonable typical measurement error, and can be used as a perturbation

The perturbation effect is thus $\frac{\sin(11°) - \sin(9°)}{2°} = \frac{0.1908 - 0.1564}{2°} = 0.0172$ per °

"For an angle of 10°, the sine function has an error sensitivity of 0.0172 per degree of error in the angle measurement."

Exercise 1: *Calculate the error sensitivity of the sine function at 80°*

Uses of error-sensitivity values

Error-sensitivity estimates can be used in several ways:

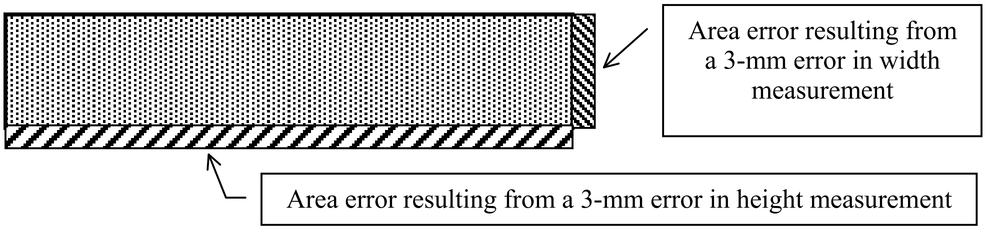
[a] *Judging the range of actual values that are consistent with a measurement.* If the solution to a problem is 100 feet multiplied by the sine of a angle measured as 10°, giving a solution of 17.36 feet, the error sensitivity calculated above implies that an error of 1.72 feet in the solution will result from each 1 degree of error in the measurement. If experience has shown that the method used to measure the angle has a stability of no better than ± ½ degree, then errors of up to 0.86 feet (= ½ degree × 1.72 feet per degree) can be expected,

and any actual solution in the range from 16.50 feet (= 17.36 – 0.86) to 18.26 feet (= 17.36 + 0.86) is consistent with the 10° angle measurement.

[b] *Figuring out in advance how accurate a measurement needs to be* in order ensure that the solution can be depended to be within a limited range. If an error of less than 3 inches (= 0.25 feet) is needed for the illustration above, for example, what does this imply about the required angle-measurement accuracy?

$$\frac{\text{0.25 feet of allowed result error}}{\text{1.72 feet of result error per degree of measurement error}} = \begin{array}{c}0.145^\circ \text{ allowed error} \\ \text{in angle measurement}\end{array}$$

[c] *Determining the biggest source of error* among several measurements used in a process. By making separate error-sensitivity estimates for each side-length measurement in a calculation of the area of a 15mm-by-75mm rectangle, for example, it can be seen that an error in measuring the short side causes 5 times as much error in the area calculation as an error or the same size in measuring the long side.



Calculations that combine two or more measurements

Many of the most interesting measurement-related calculation processes combine the values of two or more measurements to produce their result. Each measurement used will have its own errors, although measurements of the same kind may share systematic errors in some cases (such as when a short ruler is used), and sometimes the same effect (e.g., a lightning strike in the neighborhood) that causes noise in one measurement also affects other measurements in some related way. This can be a confusing situation, but several types of measurement combination are susceptible to mathematical treatment.

Measurements of the same kind: A case of particular interest is when the measurements being combined are produced by the same measurement process, such as the lengths of two sides of a rectangle both being measured with a ruler. In this case, the size of the typical errors will be about the same in both cases, so an analysis that uses the same perturbation value for both measurements is justified.

Averaging: An important instance of same-kind combinations is when the measurement of the same object is repeated with the same measurement process, with the results averaged together.

Averaging two measurements of the same kind

[a] **offset and scaling errors:** Since both measurements will have the same systematic errors, the systematic error of their average will be unchanged.

[b] **random error:** While sometimes the random errors in both measurements will be in the same direction, just as often they will be in opposite directions and will partially cancel each other. The random error of the sum will thus typically be less than twice as much as the typical errors in the measurements, although more than either of them. Since this sum is divided in half when forming the average, the typical error in the averaged value is less than in the individual measurements (about 30% less, as it turns out – see the explanation in the "Adding two measurements" section).

For averaged values of repeated measurements, the improvement in stability (which applies only to the random part of the error) is proportional to the square root of the number of repeated measurements. Thus averaging 25 measurements should reduce the random noise by a factor of 5.

$$\boxed{Noise_{result} = \sqrt{\left(Noise_A\right)^2 + \left(Noise_B\right)^2}}$$

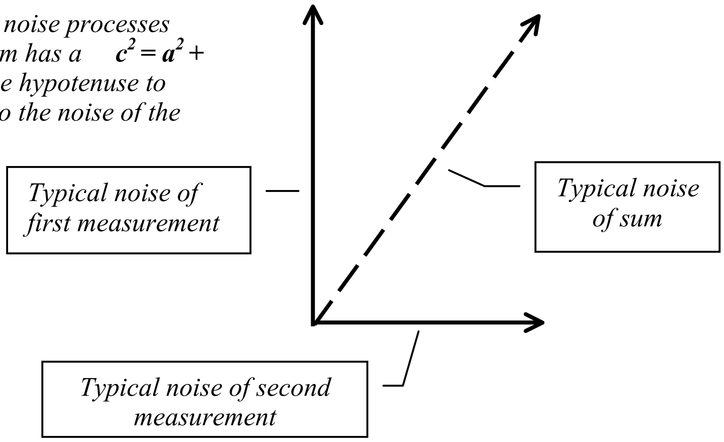


Figure 2. *When two random noise processes are added, the noise of the sum has a* $c^2 = a^2 + b^2$ *relationship (like that of the hypotenuse to the sides of a right triangle) to the noise of the*

Adding two measurements of the same kind

[a] **offset errors:** The offset error of the sum will be the sum of the offset errors of the two measurements, and thus in this case be twice the value for one measurement.

[b] **scaling errors:** If both measurements have the same scaling error factor $\boldsymbol{s}$, their sum also has that scaling error, since $\boldsymbol{s} \times \boldsymbol{a} + \boldsymbol{s} \times \boldsymbol{b} = \boldsymbol{s} \times (\boldsymbol{a} + \boldsymbol{b})$.

[c] **random errors:** While sometimes the random errors in both measurements will be in the same direction, and thus add to each other, just as often they will be in opposite directions and will partially cancel each other. As explained in the previous section, this results in the typical noise for the sum value being larger than the typical noise for either measurement but less than the sum. The exact relationship is shown in Figure 2. If both measurements have the same amount of noise, the noise of the sum will be larger by a factor of the square root of 2, or approximately 1.414.

Subtracting a measurement from another of the same kind

[a] **offset errors:** Here is a case where systematic errors cancel. Since the two measurements have the same offset error, their difference has no offset error. (This approach is often used to eliminate offset errors in practical situations.)

[b] **scaling errors:** These are the same as for addition. If both measurements have the same scaling error factor $\boldsymbol{s}$, their difference will also have that scaling error, since $\boldsymbol{s} \times \boldsymbol{a} - \boldsymbol{s} \times \boldsymbol{b} = \boldsymbol{s} \times (\boldsymbol{a} - \boldsymbol{b})$.

[c] **random errors:** The noise in the difference of two measurements is computed in exactly the same way as that for a sum. *Therefore the noise of the difference of two measurements is more than the noise of either measurement.* Since a difference may be much smaller than the measurements from which it was formed, it often happens that the relative noise of a difference is large enough to be a problem even though the noise did not seem all that significant relative to the original measurements.

HOMEWORK – MEASUREMENT-ERROR SENSITIVITY

Work all problems on a separate sheet of paper. Show all your work.

[1] Calculate the error sensitivity of the tangent function at 30°.

[2] Calculate the sensitivity of area calculations to error in side length, for a square with a side length of 45 millimeters.

[3] A rectangle has a height of 10 mm and a width of 80 mm.

[a] Calculate the sensitivity of area calculations to error in height measurement.

[b] Calculate the sensitivity of area calculations to error in width measurement.

[4] If a length-estimation process has an error sensitivity of 28.4 mm per degree, what error in the result is expected from an error in angle measurement of 1.3 degrees?

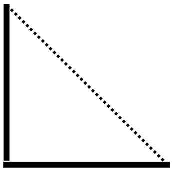
[5] If an area-estimation process based on side length has an error sensitivity of 12.3 in^2 per inch, how much error in area is expected if an error of 0.63 inches has been made in side length?

[6] If a the length of a beam is proportional to the tangent of an angle that is measured to be 30°, what percentage error in beam length is expected if there is a 2.1° error in the angle measurement? [Hint: you have already computed the sensitivity in problem 1.]

[7] The size of tiles with intended areas of 2.56 in^2 are checked by measurement of the length of their sides.

[a] Compute the sensitivity of the area measurement to error in side length.

[b] If the typical error for the side-measurement process is ± 0.12 inches, is a measured value of 1.54 inches for a side length consistent with the intended area?



[8] To check if the angle in a corner is a right angle, the length of the hypotenuse is measured at a distance of exactly 300 mm from the corner on each side. The measured value was 423 mm, using a measurement process that has been found to have a typical error of ± 2 mm. Is this consistent with the angle being a right angle?

[9] If a measurement process produces measurements with a typical random error of ± 3.5 mm, what is the expected typical random error for averages of 100 such measurements?

[10] If a process produces measurements with a typical random error of 3% of the reference value, what relative random error would be expected for averages of 25 such measurements?

[11] If a process produces measurements with a typical random error of ± 1.3 pounds, what random error would be expected for averages of 10 such measurements?

[12] If a measurement process has a random noise of 4% for a certain measurement, how many measurements should be averaged to reduce the random noise to 1/2 %?

[13] If two measurements from a process with a typical random error of ± 1.3 pounds are added, what is the expected random error for the sum?

[14] The height of a West-Texas hill is measured by using an air-pressure altimeter to measure height above sea level at the base of the hill (where the reading was 3627 feet above sea level) and at its summit (where the reading was 3745 feet above sea level).

[a] What is the estimated height of the hill (from its base to its summit)?

[b] What would be the effect on the accuracy of the hill-height estimate if the altimeter has an offset error of +200 feet?

[c] What would be the effect on the accuracy of that estimate if the altimeter has a scaling error of +10% (that is, all its readings are 110% of the true values)?

[d] If the altimeter has a typical random error of ± 50 feet, what effect does that have on the accuracy of the hill-height estimate?

DISTRIBUTION PATTERNS OF RANDOM MEASUREMENTS

Describing measurement stability: While it is not possible to fully correct for the small random variations that are part of all measurement processes, the impact of those errors can be minimized by computing statistics that characterize the expected average value and the stability of the processes that provide measurements we use. These statistics give us information about how much we should depend on individual measurements and, by averaging, provide a method of reducing the noise in estimates based on measurements.

Exercise 1: Compute these statistics for the mileage (or other) measurements you produced for problem 3 of the Day 8 homework.

[a] Compute distance measurements by subtracting two appropriate odometer readings.

[b] The mean value of the distance measurements is ____________ miles.

[c] The range of the distance measurements is _______________ miles.

[d] The greatest deviation from the mean is _________________ miles.

[e] The average deviation from the mean is _________________ miles.

[f] What average random error do these results imply for each odometer reading? [Hint: the distance measurements are a combination of two odometer readings.]

The sources of measurement variation: In most measurement situations, the random variation seen in repeated measurements results from the addition of many minor effects, each making a small addition or subtraction from the value. Such small effects, which will cancel in some cases and add together in others, result in processes where repeated measurements are distributed in a "bell-shaped" pattern in which values close to the average are most common. Values become less common the further they are from the average, but still can occasionally (when almost all the small effects are going the same way) end up rather far away from the typical values.

Exercise 2: The coin-flip data you produced for problem 2 in the Day 8 homework can to used to show how partial cancellation results when random effects are added. The number of heads in each set of 4 coin flips can be one of five values: 0, 1, 2, 3, or 4. Contribute the results of the 3 trials you made to the tabulation being accumulated by the teacher, then write the results of that whole-class tabulation below in the table provided, then compute the percentage that each number-of-heads result is of the total number of recorded sets of four coin flips:

	0 heads	1 head	2 heads	3 heads	4 heads
Instances of this case					
% of overall total					

Note that the "average" 2-heads case is the most common, and that the far-from-average 0-heads and 4-heads cases are least common, with the near-average 1-head and 3-heads cases happening with intermediate frequency. (The reason that these printed materials can predict in advance the basic shape of the distribution that will result from the homework assignments of your class is that the *statistics* associated with a large number of random events can be predicted reasonably well even though the individual events themselves are unpredictable.)

The basic cause of the coin-flip results is that there are more ways in which random events can combine to give values close to the average than far away. Figure 1 shows the full set of 16 possible results for a sequence of four coin flips. Thus 37.5% of the possible combinations will result in the average value of 2 heads, 25% of the possibilities have 3 heads and another 25% have 1 head, while the extreme cases of 4 heads and 0 heads each represent only 6.25% of the possibilities.

Figure 1
Possible different
results for 4 flips

HHHH – *4 heads*

HHHT
HHTH *3 heads*
HTHH *(4 ways)*
THHH

HHTT
HTHT
HTTH *2 heads*
THHT *(6 ways)*
THTH
TTHH

HTTT
THTT *1 head*
TTHT *(4 ways)*
TTTH

TTTT – *0 heads*

The percentage results from the tabulation of your homework exercise will not match these values exactly, because the random flipping process does not go through this list in any dependable sequence. However, as the number of instances of a random process becomes larger, the difference between results and possibility percentages become smaller.

The diagrams in Figure 2 give graphical representations of the sets of possibilities for two, four, and eight coins. Notice that the number of possible combinations increases rapidly, with the eight-coin case having 256 variations. The numbers below each column are the approximate percentages of the total possibilities that the column represents.

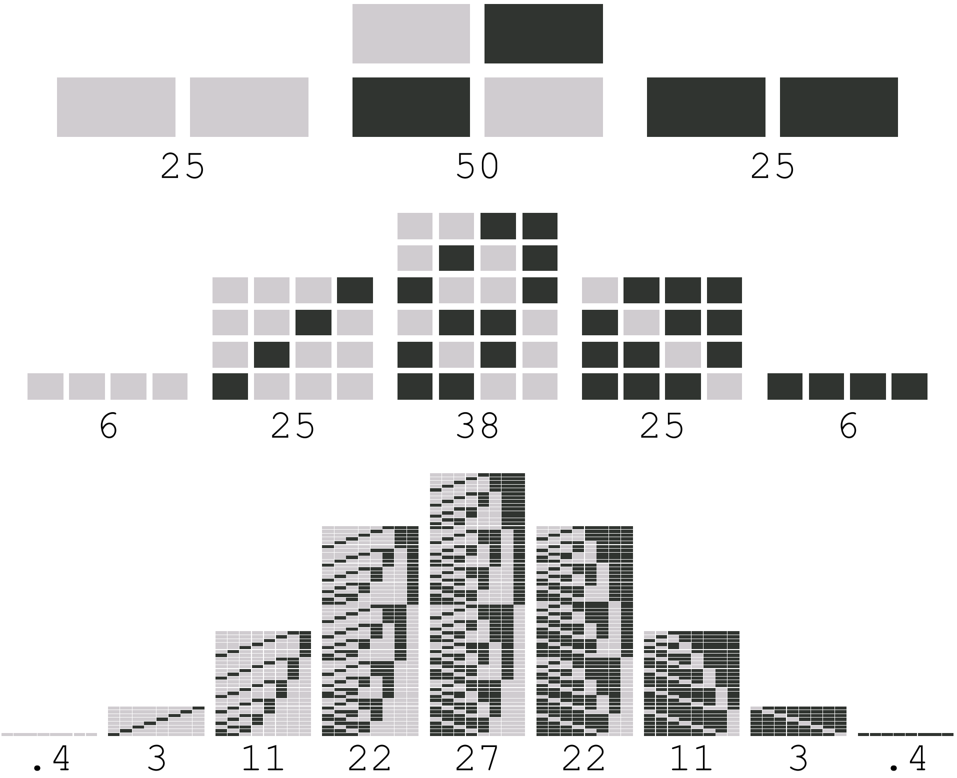


Figure 2. Distribution of possible different results for 2,4, and 6 coin flips