APPENDIX C-4: GRAPHS & ERROR SENSITIVITY

Graphical Interpretation of Error Sensitivity

Any process by which a single measured value is used to compute some result (such as computing a square's area from a measurement of its side length) can be represented as a mathematical graph in which the horizontal direction corresponds to different values of the measured quantity and the vertical direction corresponds to the values that result from the computation.

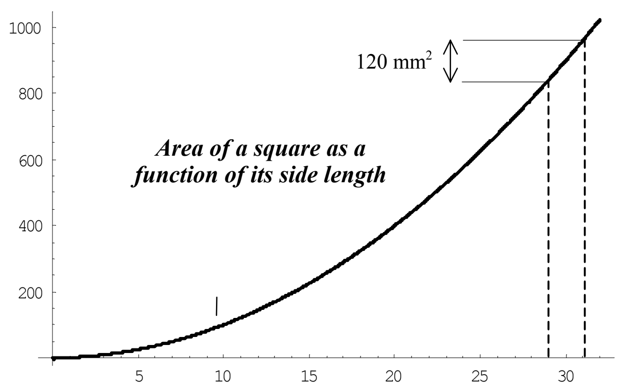


Figure 1: The heavy solid line is a graph of the ***area*** of a square (vertical axis) as a function of the ***length of its side*** (horizontal axis). Note that the same amount of error in side-length measurement will cause a bigger error in area for a larger side.

The *error sensitivity* of this computation process at a reference value is the ***slope*** of the graph at that value, which can be approximated by the ratio of the change in result to the change in measurement.

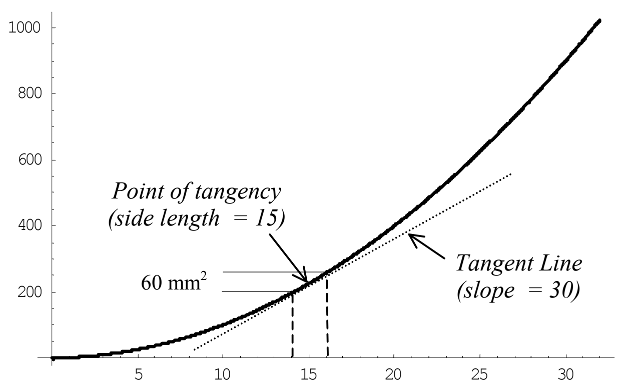
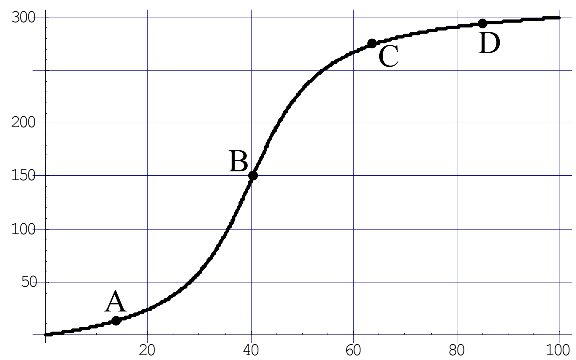


Figure 2: Another way of thinking about the slope of a graph is to draw the "tangent" line that touches (but usually does not cross) the graph at the value of interest. ("Tangent" comes from a Latin word for "touching".)

The dotted line shows such a tangent line at the point corresponding to a side length of 15 on the same graph as Figure 1. The definition of the slope of a graph at a point is the slope of the tangent line at that point.

When a graph of the function used to compute a result from a measurement is available, several types of questions about error sensitivity can be answered by inspection of the graph, and others can often be approximately answered by drawing a tangent line and measuring its slope. The thing to remember is that the slope of the function (that is, the ratio of the change in result to the change in measurement, in the immediate area of the reference point) is equal to the error sensitivity.

Example 1: *The graph shows how a computational result depends on a measurement value.*



[a] Approximately which measurement and result values are indicated by point A?

[b] At which of the four labeled points does this process have the greatest error sensitivity?

[c] At which of the marked points is this process least sensitive to measurement error?

[d] Approximately what is the slope at point B?

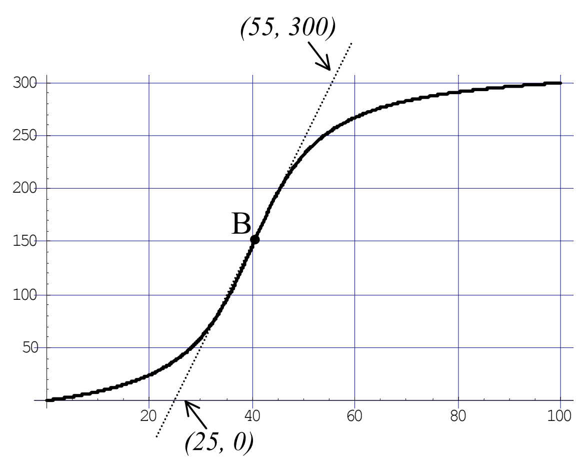
Answers to Example 1:

[a] ***measurement ≅ 26, result ≅ 42***. *[Estimate the values from the graph. The measurement corresponds to the horizontal position of the point and the result to the vertical position.]*

[b] ***Point B***. *[Since the error sensitivity of the process is equal to the slope of the graph at the corresponding point, the greatest error sensitivity will be at the point at which the graph has the greatest (that is, the steepest) slope. Of the points labeled, the slope is steepest at B.]*

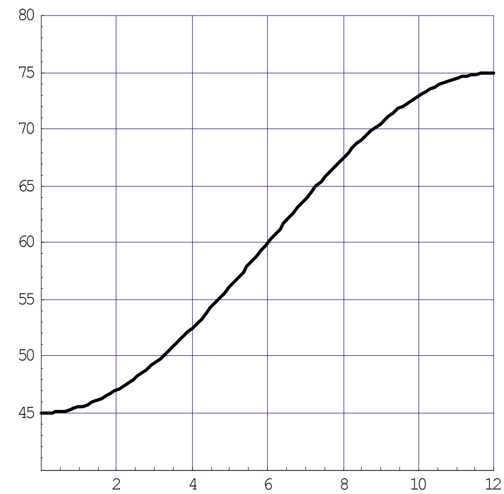
[c] ***Point D***. *[Of the points marked, the slope at D is lowest (the graph is rising most slowly).]*

[d] ***slope ≅ 10*** *[First draw the tangent line through B (dotted line), extending it so that it intersects the top and bottom grid lines. Those lines are for results of 300 and 0, respectively, so the difference in result is 300. Next estimate from the drawn line the horizontal positions of the top intersection point (about 55) and the bottom one (about 25), giving a difference of about 30. The slope is "rise divided by run", which in this case is thus about 10. Notice that since the vertical and horizontal scales are different (300 on the vertical scale is the same distance as about 60 on the horizontal scale), you have to use the printed scale numbers rather than direct measurements to determine the values for rise and run.]*



Exercise 1:

The graph to the right shows the temperature for a particular day measured at times from sunrise until 12 hours later.



[a] At how many hours after sunrise would a delay in when the temperature was measured make the biggest difference in its value?

[b] About how fast was the temperature changing at that time?

[c] How big an error in temperature could a 15-minute mistake in measurement time cause?

[d] At what times would a delay in the time of measurement not make much difference in the measured temperature value?

The graph used in Exercise 1 is a piece of an important kind of function that is the basis of many repetitive processes, and is also related to the sine and cosine ratios that we used in analyzing right triangles. We will examine these *sinusoidal* functions in more detail later, but here we will just identify which parts of them match the shape of the sine and cosine functions between 0 and 90 degrees. Notice that a function of this kind changes slowly at its maximum and minimum values (where the slope of the tangent line is zero) and changes most rapidly halfway in-between.

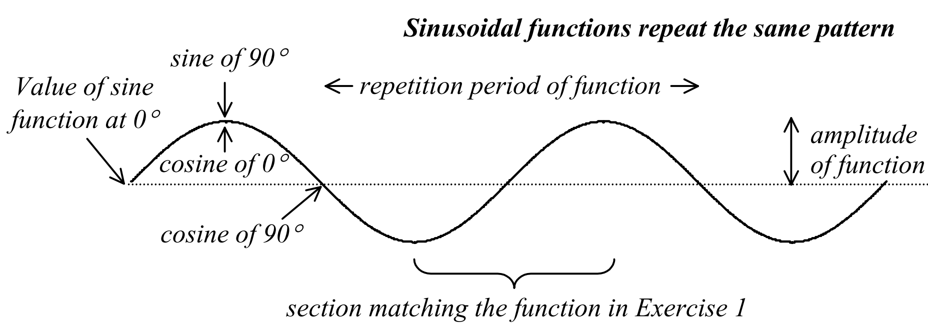


Figure 3: Sinusoidal functions

Graph of the *tangent* trigonometric ratio:

While the sine and cosine functions change value gradually and smoothly as the angle changes, the tangent function of an angle has more dramatic changes, especially when the angle gets close to 90°. This is because it is possible for the denominator of the ratio to become much smaller than the numerator, resulting in a large quotient. In such cases, small errors in the angle can cause very large errors in the result.

Example 2:

A searchlight mounted 100 feet high on a tower is used to look at the surrounding flat and level plain. The angle between the tower and the path of the searchlight can be varied from 0° (straight down at the base of the tower) to 90° (straight sideways, parallel to the ground).

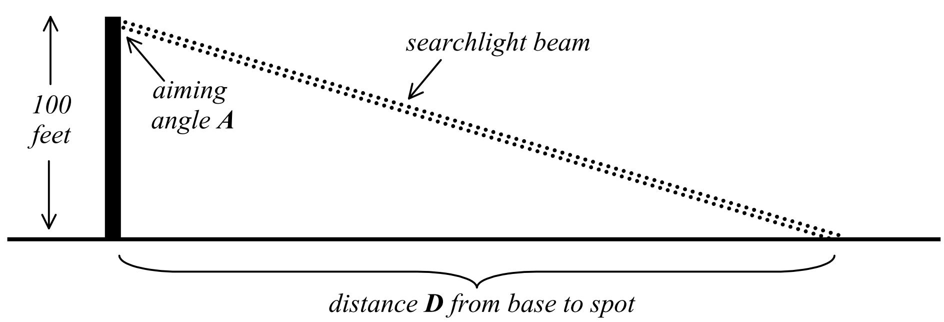
[a] How far away from the base of the tower will the spot of light on the ground be for angle settings of 0°, 20°, 40°, 60°, and 80°?

[b] Plot a graph of the values computed in [a], using a appropriate vertical scale that includes all the values.

[c] What will happen for an angle setting of 90°?

[d] Compute the distances for angle settings from 81° to 89° in one-degree steps.

[e] Compute the distances for angles of 89.9°, 89.99°, and 89.999°.



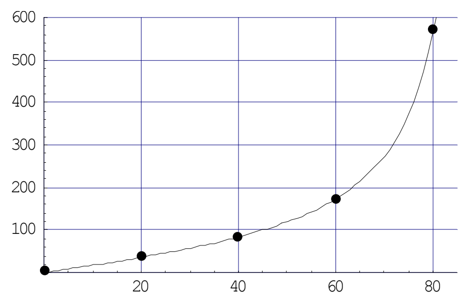
$$\tan(A) = \frac{\text{distance to spot}}{\text{height of tower}} = \frac{D}{100\,ft} \qquad \text{Therefore, } D = \tan(A) \times 100\,ft$$

Answers to Example 2:

[a]

Angle *A*	0°	20°	40°	60°	80°
tan(*A*)	0.0000	0.3639	0.8391	1.7321	5.6713
Distance *D*	0.00 ft	36.39 ft	83.91 ft	173.21 ft	567.13 ft

[b]



The tangent function rises rapidly as the angle gets closer to 90°, which means that results depending on tangent computations then become much more sensitive to error in angle measurement.

[c] At 90°, the path of the searchlight rays will not intersect the ground. *[This is related to the fact that computing the tangent of 90° would require dividing by zero, which is meaningless. Thus the tangent function does not have a value at 90°.]*

[d]

Angle *A*	81°	82°	83°	84°	85°	86°	87°	88°	89°
tan(*A*)	6.31	7.12	8.14	9.51	11.43	14.30	19.08	28.64	57.29
D (ft)	631	712	814	951	1143	1430	1908	2864	5729

[e] tan(89.9°) ≅ 572.96, so D ≅ 57,296 feet

tan(89.99°) ≅ 5729.58, so D ≅ 572,958 feet

tan(89.999°) ≅ 57295.78, so D ≅ 5,729,578 feet

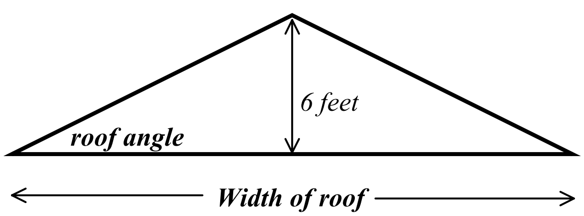
Exercise 2:

The peak of a simple roof (see diagram below) is fixed at 6 feet above the ceiling level, and the horizontal width is cut to whatever it takes to make the desired angle for the roof.

[a] Compute what width of roof corresponds to angles of 15°, 30°, 45°, 60°, and 75°.

[b] Using the values determined in [a], draw a graph showing the relationship between roof angle and width for this roof.

[c] For which of the angles listed in [a] would making a 1° mistake in the angle value used cause the most error in width?



*(notice that the width is **twice** the length of one of the right triangles formed by the altitude)*

HOMEWORK – Graphical Interpretation of Error Sensitivity

Work all problems on a separate sheet of paper. Show all your work.

[1] The area of an equilateral triangle (where all three sides are equal in length) is

$$\text{Area} = \frac{\sqrt{3}}{4}s^2 \cong 0.433\ s^2$$

[a] Compute the area for side lengths of 0, 2, 4, 6, 8, and 10 mm.

[b] Graph this function between 0 and 10 mm by plotting the values

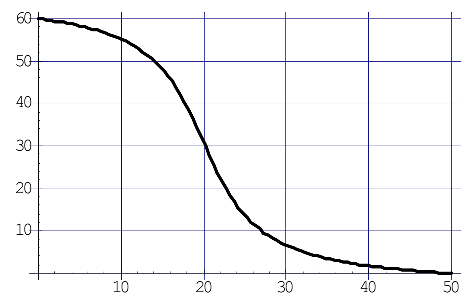
[c] Draw the tangent line at s = 6 mm.

[d] From the tangent line, estimate the slope of the graph at that point.

[2] The graph on the right shows the results of a computation (vertical scale) based on a measurement (horizontal scale).

[a] At about what measurement value is the computation most sensitive to error in the measurement?

[b] Estimate the error sensitivity at that point.



[3] The diagram to the right has graphs of the sine function and the tangent function from 0 to 60 degrees. The graphs are overlaid on each other and graphed with the same scales.

[a] At 50°, which function has greater sensitivity to errors in angle measurement?

[b] Estimate the error sensitivity of each function at 50° by drawing the tangent lines and estimating their slope.

[c] The graph suggests that in this range the tangent function is always greater than the sine function of the same angle. Can you explain why this is true? ***[Hint: use the definitions of the sine and tangent ratios.]***